

# Mathematical Programming: An Overview

---

# 1

Management science is characterized by a scientific approach to managerial decision making. It attempts to apply mathematical methods and the capabilities of modern computers to the difficult and unstructured problems confronting modern managers. It is a young and novel discipline. Although its roots can be traced back to problems posed by early civilizations, it was not until World War II that it became identified as a respectable and well defined body of knowledge. Since then, it has grown at an impressive pace, unprecedented for most scientific accomplishments; it is changing our attitudes toward decision-making, and infiltrating every conceivable area of application, covering a wide variety of business, industrial, military, and public-sector problems.

Management science has been known by a variety of other names. In the United States, *operations research* has served as a synonym and it is used widely today, while in Britain *operational research* seems to be the more accepted name. Some people tend to identify the scientific approach to managerial problem-solving under such other names as systems analysis, cost-benefit analysis, and cost-effectiveness analysis. We will adhere to *management science* throughout this book.

Mathematical programming, and especially linear programming, is one of the best developed and most used branches of management science. It concerns the optimum allocation of limited resources among competing activities, under a set of constraints imposed by the nature of the problem being studied. These constraints could reflect financial, technological, marketing, organizational, or many other considerations. In broad terms, mathematical programming can be defined as a mathematical representation aimed at programming or planning the best possible allocation of scarce resources. When the mathematical representation uses linear functions exclusively, we have a linear-programming model.

In 1947, George B. Dantzig, then part of a research group of the U.S. Air Force known as Project SCOOP (Scientific Computation Of Optimum Programs), developed the *simplex method* for solving the general linear-programming problem. The extraordinary computational efficiency and robustness of the simplex method, together with the availability of high-speed digital computers, have made linear programming the most powerful optimization method ever designed and the most widely applied in the business environment.

Since then, many additional techniques have been developed, which relax the assumptions of the linear-programming model and broaden the applications of the mathematical-programming approach. It is this spectrum of techniques and their effective implementation in practice that are considered in this book.

## 1.1 AN INTRODUCTION TO MANAGEMENT SCIENCE

Since mathematical programming is only a tool of the broad discipline known as management science, let us first attempt to understand the management-science approach and identify the role of mathematical programming within that approach.

It is hard to give a noncontroversial definition of management science. As we have indicated before, this is a rather new field that is renewing itself and changing constantly. It has benefited from contributions originating in the social and natural sciences, econometrics, and mathematics, much of which escape the rigidity of a definition. Nonetheless it is possible to provide a general statement about the basic elements of the management-science approach.

Management science is characterized by the use of *mathematical models* in providing guidelines to managers for making effective decisions within the state of the current information, or in seeking further information if current knowledge is insufficient to reach a proper decision.

There are several elements of this statement that are deserving of emphasis. First, the essence of management science is the model-building approach—that is, an attempt to capture the most significant features of the decision under consideration by means of a mathematical abstraction. Models are simplified representations of the real world. In order for models to be useful in supporting management decisions, they have to be simple to understand and easy to use. At the same time, they have to provide a complete and realistic representation of the decision environment by incorporating all the elements required to characterize the essence of the problem under study. This is not an easy task but, if done properly, it will supply managers with a formidable tool to be used in complex decision situations.

Second, through this model-design effort, management science tries to provide guidelines to managers or, in other words, to increase managers' understanding of the consequences of their actions. There is never an attempt to replace or substitute for managers, but rather the aim is to *support* management actions. It is critical, then, to recognize the strong interaction required between managers and models. Models can expediently and effectively account for the many interrelationships that might be present among the alternatives being considered, and can explicitly evaluate the economic consequences of the actions available to managers within the constraints imposed by the existing resources and the demands placed upon the use of those resources. Managers, on the other hand, should formulate the basic questions to be addressed by the model, and then interpret the model's results in light of their own experience and intuition, recognizing the model's limitations. The complementarity between the superior computational capabilities provided by the model and the higher judgmental capabilities of the human decision-maker is the key to a successful management-science approach.

Finally, it is the complexity of the decision under study, and not the tool being used to investigate the decision-making process, that should determine the amount of information needed to handle that decision effectively. Models have been criticized for creating unreasonable requirements for information. In fact, this is not necessary. Quite to the contrary, models can be constructed within the current state of available information and they can be used to evaluate whether or not it is economically desirable to gather additional information.

The subject of proper model design and implementation will be covered in detail in Chapter 5.

## 1.2 MODEL CLASSIFICATION

The management-science literature includes several approaches to classifying models. We will begin with a categorization that identifies broad types of models according to the degree of realism that they achieve in representing a given problem. The model categories can be illustrated as shown in Fig. 1.1.

### Operational Exercise

The first model type is an *operational exercise*. This modeling approach operates directly with the real environment in which the decision under study is going to take place. The modeling effort merely involves designing a set of experiments to be conducted in that environment, and measuring and interpreting the results of those experiments. Suppose, for instance, that we would like to determine what mix of several crude oils should be blended in a given oil refinery to satisfy, in the most effective way, the market requirements for final products to be delivered from that refinery. If we were to conduct an operational exercise to support that decision, we would try different quantities of several combinations of crude oil types directly in the actual

refinery process, and observe the resulting revenues and costs associated with each alternative mix. After performing quite a few trials, we would begin to develop an understanding of the relationship between the

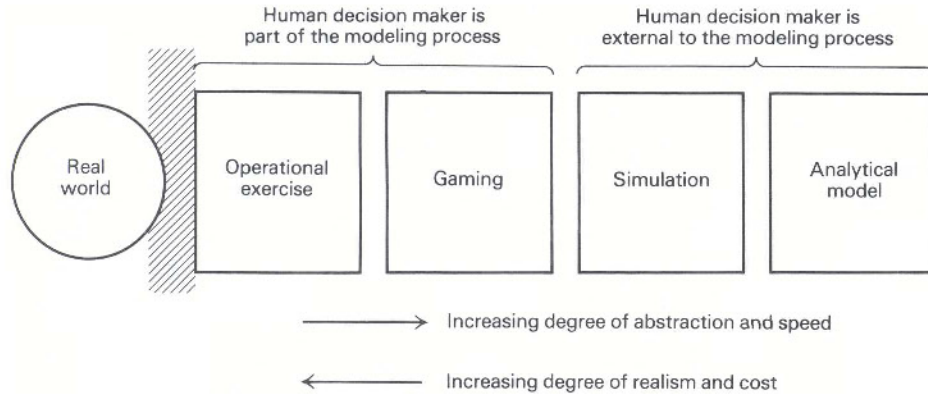


Fig. 1.1 Types of model representation.

crude oil input and the net revenue obtained from the refinery process, which would guide us in identifying an appropriate mix.

In order for this approach to operate successfully, it is mandatory to design experiments to be conducted carefully, to evaluate the experimental results in light of errors that can be introduced by measurement inaccuracies, and to draw inferences about the decisions reached, based upon the limited number of observations performed. Many statistical and optimization methods can be used to accomplish these tasks properly. The essence of the operational exercise is an inductive learning process, characteristic of empirical research in the natural sciences, in which generalizations are drawn from particular observations of a given phenomenon.

Operational exercises contain the highest degree of realism of any form of modeling approach, since hardly any external abstractions or oversimplifications are introduced other than those connected with the interpretation of the observed results and the generalizations to be drawn from them. However, the method is exceedingly, usually prohibitively, expensive to implement. Moreover, in most cases it is impossible to exhaustively analyze the alternatives available to the decision-maker. This can lead to severe *suboptimization* in the final conclusions. For these reasons, operational exercises seldom are used as a pure form of modeling practice. It is important to recognize, however, that direct observation of the actual environment underlies most model conceptualizations and also constitutes one of the most important sources of data. Consequently, even though they may not be used exclusively, operational exercises produce significant contributions to the improvement of managerial decision-making.

### Gaming

The second type of model in this classification is *gaming*. In this case, a model is constructed that is an abstract and simplified representation of the real environment. This model provides a responsive mechanism to evaluate the effectiveness of proposed alternatives, which the decision-maker must supply in an organized and sequential fashion. The model is simply a device that allows the decision-maker to test the performance of the various alternatives that seem worthwhile to pursue. In addition, in a gaming situation, all the human interactions that affect the decision environment are allowed to participate actively by providing the inputs they usually are responsible for in the actual realization of their activities. If a gaming approach is used in our previous example, the refinery process would be represented by a computer or mathematical model, which could assume any kind of structure.

The model should reflect, with an acceptable degree of accuracy, the relationships between the inputs and outputs of the refinery process. Subsequently, all the personnel who participate in structuring the decision process in the management of the refinery would be allowed to interact with the model. The production manager would establish production plans, the marketing manager would secure contracts and develop marketing

strategies, the purchasing manager would identify prices and sources of crude oil and develop acquisition programs, and so forth. As before, several combinations of quantities and types of crude oil would be tried, and the resulting revenues and cost figures derived from the model would be obtained, to guide us in formulating an optimal policy. Certainly, we have lost some degree of realism in our modeling approach with respect to the operational exercise, since we are operating with an abstract environment, but we have retained some of the human interactions of the real process. However, the *cost of processing each alternative* has been reduced, and the *speed of measuring the performance* of each alternative has been increased.

Gaming is used mostly as a learning device for developing some appreciation for those complexities inherent in a decision-making process. Several management games have been designed to illustrate how marketing, production, and financial decisions interact in a competitive economy.

### Simulation

*Simulation* models are similar to gaming models except that all human decision-makers are removed from the modeling process. The model provides the means to evaluate the performance of a number of alternatives, supplied externally to the model by the decision-maker, without allowing for human interactions at intermediate stages of the model computation.

Like operational exercises and gaming, simulation models neither generate alternatives nor produce an optimum answer to the decision under study. These types of models are inductive and empirical in nature; they are useful only to assess the performance of alternatives identified previously by the decision-maker.

If we were to conduct a simulation model in our refinery example, we would program in advance a large number of combinations of quantities and types of crude oil to be used, and we would obtain the net revenues associated with each alternative without any external inputs of the decision-makers. Once the model results were produced, new runs could be conducted until we felt that we had reached a proper understanding of the problem on hand.

Many simulation models take the form of computer programs, where logical arithmetic operations are performed in a prearranged sequence. It is not necessary, therefore, to define the problem exclusively in analytic terms. This provides an added flexibility in model formulation and permits a high degree of realism to be achieved, which is particularly useful when uncertainties are an important aspect of the decision.

### Analytical Model

Finally, the fourth model category proposed in this framework is the *analytical model*. In this type of model, the problem is represented completely in mathematical terms, normally by means of a criterion or objective, which we seek to maximize or minimize, subject to a set of mathematical constraints that portray the conditions under which the decisions have to be made. The model computes an optimal solution, that is, one that satisfies all the constraints and gives the best possible value of the objective function.

In the refinery example, the use of an analytical model implies setting up as an objective the *maximization of the net revenues* obtained from the refinery operation as a function of the types and quantities of the crude oil used. In addition, the technology of the refinery process, the final product requirements, and the crude oil availabilities must be represented in mathematical terms to define the constraints of our problem. The solution to the model will be the exact amount of each available crude-oil type to be processed that will maximize the net revenues within the proposed constraint set. Linear programming has been, in the last two decades, the indisputable analytical model to use for this kind of problem.

Analytical models are normally the least expensive and easiest models to develop. However, they introduce the highest degree of simplification in the model representation. As a rule of thumb, it is better to be as much to the right as possible in the model spectrum (no political implication intended!), provided that the resulting degree of realism is appropriate to characterize the decision under study.

Most of the work undertaken by management scientists has been oriented toward the development and implementation of analytical models. As a result of this effort, many different techniques and methodologies have been proposed to address specific kinds of problems. Table 1.1 presents a classification of the most important types of analytical and simulation models that have been developed.



**Table 1.1** Classification of Analytical and Simulation Models

	<i>Strategy evaluation</i>	<i>Strategy generation</i>
<i>Certainty</i>	Deterministic simulation Econometric models Systems of simultaneous equations Input-output models	Linear programming Network models Integer and mixed-integer programming Nonlinear programming Control theory
<i>Uncertainty</i>	Monte Carlo simulation Econometric models Stochastic processes Queueing theory Reliability theory	Decision theory Dynamic programming Inventory theory Stochastic programming Stochastic control theory

Statistics and subjective assessment are used in all models to determine values for parameters of the models and limits on the alternatives.

The classification presented in Table 1.1 is not rigid, since strategy evaluation models are used for improving decisions by trying different alternatives until one is determined that appears “best.” The important distinction of the proposed classification is that, for strategy evaluation models, the user must first choose and construct the alternative and then evaluate it with the aid of the model. For strategy generation models, the alternative is not completely determined by the user; rather, the class of alternatives is determined by establishing constraints on the decisions, and then an algorithmic procedure is used to automatically generate the “best” alternative within that class. The horizontal classification should be clear, and is introduced because the inclusion of uncertainty (or not) generally makes a substantial difference in the type and complexity of the techniques that are employed. Problems involving uncertainty are inherently more difficult to formulate well and to solve efficiently.

This book is devoted to mathematical programming—a part of management science that has a common base of theory and a large range of applications. Generally, mathematical programming includes all of the topics under the heading of strategy generation except for decision theory and control theory. These two topics are entire disciplines in themselves, depending essentially on different underlying theories and techniques. Recently, though, the similarities between mathematical programming and control theory are becoming better understood, and these disciplines are beginning to merge. In mathematical programming, the main body of material that has been developed, and more especially applied, is under the assumption of certainty. Therefore, we concentrate the bulk of our presentation on the topics in the upper righthand corner of Table 1.1. The critical emphasis in the book is on developing those principles and techniques that lead to *good formulations* of actual decision problems and *solution procedures* that are efficient for solving these formulations.

### 1.3 FORMULATION OF SOME EXAMPLES

In order to provide a preliminary understanding of the types of problems to which mathematical programming can be applied, and to illustrate the kind of rationale that should be used in formulating linear-programming problems, we will present in this section three highly simplified examples and their corresponding linear-programming formulations.

**Charging a Blast Furnace\*** An iron foundry has a firm order to produce 1000 pounds of castings containing at least 0.45 percent manganese and between 3.25 percent and 5.50 percent silicon. As these particular castings are a special order, there are no suitable castings on hand. The castings sell for \$0.45 per pound. The foundry has three types of pig iron available in essentially unlimited amounts, with the following properties:

\* Excel spreadsheet available at [http://web.mit.edu/15.053/www/Sect1.3\\_Blast\\_Furnace.xls](http://web.mit.edu/15.053/www/Sect1.3_Blast_Furnace.xls)

	<i>Type of pig iron</i>		
	A	B	C
Silicon	4 %	1 %	0.6%
Manganese	0.45%	0.5%	0.4%

Further, the production process is such that pure manganese can also be added directly to the melt. The costs of the various possible inputs are:

Pig A	\$21/thousand pounds
Pig B	\$25/thousand pounds
Pig C	\$15/thousand pounds
Manganese	\$ 8/pound.

It costs 0.5 cents to melt down a pound of pig iron. Out of what inputs should the foundry produce the castings in order to maximize profits?

The first step in formulating a linear program is to define the *decision variables* of the problem. These are the elements under the control of the decision-maker, and their values determine the solution of the model. In the present example, these variables are simple to identify, and correspond to the number of pounds of pig A, pig B, pig C, and pure manganese to be used in the production of castings. Specifically, let us denote the decision variables as follows:

- $x_1$  = Thousands of pounds of pig iron A,
- $x_2$  = Thousands of pounds of pig iron B,
- $x_3$  = Thousands of pounds of pig iron C,
- $x_4$  = Pounds of pure manganese.

The next step to be carried out in the formulation of the problem is to determine the criterion the decision-maker will use to evaluate alternative solutions to the problem. In mathematical-programming terminology, this is known as the *objective function*. In our case, we want to maximize the total profit resulting from the production of 1000 pounds of castings. Since we are producing exactly 1000 pounds of castings, the total income will be the selling price per pound times 1000 pounds. That is:

$$\text{Total income} = 0.45 \times 1000 = 450.$$

To determine the total cost incurred in the production of the alloy, we should add the melting cost of \$0.005/pound to the corresponding cost of each type of pig iron used. Thus, the relevant unit cost of the pig iron, in dollars per thousand pounds, is:

Pig iron A	$21 + 5 = 26,$
Pig iron B	$25 + 5 = 30,$
Pig iron C	$15 + 5 = 20.$

Therefore, the total cost becomes:

$$\text{Total cost} = 26x_1 + 30x_2 + 20x_3 + 8x_4, \quad (1)$$

and the total profit we want to maximize is determined by the expression:

$$\text{Total profit} = \text{Total income} - \text{Total cost}.$$

Thus,

$$\text{Total profit} = 450 - 26x_1 - 30x_2 - 20x_3 - 8x_4. \quad (2)$$

It is worthwhile noticing in this example that, since the amount of castings to be produced was fixed in advance, equal to 1000 pounds, the maximization of the total profit, given by Eq. (2), becomes completely equivalent to the minimization of the total cost, given by Eq. (1).

We should now define the *constraints* of the problem, which are the restrictions imposed upon the values of the decision variables by the characteristics of the problem under study. First, since the producer does not want to keep any supply of the castings on hand, we should make the total amount to be produced exactly equal to 1000 pounds; that is,

$$1000x_1 + 1000x_2 + 1000x_3 + x_4 = 1000. \quad (3)$$

Next, the castings should contain at least 0.45 percent manganese, or 4.5 pounds in the 1000 pounds of castings to be produced. This restriction can be expressed as follows:

$$4.5x_1 + 5.0x_2 + 4.0x_3 + x_4 \geq 4.5. \quad (4)$$

The term  $4.5x_1$  is the pounds of manganese contributed by pig iron A since each 1000 pounds of this pig iron contains 4.5 pounds of manganese. The  $x_2$ ,  $x_3$ , and  $x_4$  terms account for the manganese contributed by pig iron B, by pig iron C, and by the addition of pure manganese.

Similarly, the restriction regarding the silicon content can be represented by the following inequalities:

$$40x_1 + 10x_2 + 6x_3 \geq 32.5, \quad (5)$$

$$40x_1 + 10x_2 + 6x_3 \leq 55.0. \quad (6)$$

Constraint (5) establishes that the minimum silicon content in the castings is 3.25 percent, while constraint (6) indicates that the maximum silicon allowed is 5.5 percent.

Finally, we have the obvious nonnegativity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

If we choose to minimize the total cost given by Eq. (1), the resulting linear programming problem can be expressed as follows:

$$\begin{aligned} \text{Minimize } z = & 26x_1 + 30x_2 + 20x_3 + 8x_4, \\ \text{subject to:} & \\ & 1000x_1 + 1000x_2 + 1000x_3 + x_4 = 1000, \\ & 4.5x_1 + 5.0x_2 + 4.0x_3 + x_4 \geq 4.5, \\ & 40x_1 + 10x_2 + 6x_3 \geq 32.5, \\ & 40x_1 + 10x_2 + 6x_3 \leq 55.0, \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0. \end{aligned}$$

Often, this algebraic formulation is represented in *tableau form* as follows:

	<i>Decision variables</i>				<i>Relation</i>	<i>Requirements</i>
	$x_1$	$x_2$	$x_3$	$x_4$		
Total lbs.	1000	1000	1000	1	=	1000
Manganese	4.5	5	4	1	$\geq$	4.5
Silicon lower	40	10	6	0	$\geq$	32.5
Silicon upper	40	10	6	0	$\leq$	55.0
Objective	26	30	20	8	=	$z$ (min)
(Optimal solution)	0.779	0	0.220	0.111		25.54

The bottom line of the tableau specifies the *optimal solution* to this problem. The solution includes the values of the decision variables as well as the minimum cost attained, \$25.54 per thousand pounds; this solution was generated using a commercially available linear-programming computer system. The underlying solution procedure, known as the *simplex method*, will be explained in Chapter 2.