Second course: Primal simplex method

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The initial simplex method is a systematic repeated procedure until it finally reaches the desired or optimal solution. The set of steps that is repeated each time in such a process is called an iteration.

Basics of the primal simplex method

In an algebraic method, working with equal equations is far simpler than inequalities. Hence, the first step of the simplex method (omit the primal for the rest of manuscript) is to convert functional constraints from an unequal form to equal one. The constraints of negativity can be left unequal because they do not enter directly into the solution process. The conversion of an inequality to equality is done by introducing slack variables. Next, we solve the following model using Simplex method in the form of an example. The model is as follows:

$(P1)\ Max\ Z = 3x_1 + 5x_2$
$s\,t.$

$\quad (1)\quad x_1 \qquad\quad \le 4$
$\quad (2)\qquad\quad 2x_2 \le 12$
$\quad (3)\quad 3x_1 + 2x_2 \le 18$
$\qquad\quad x_1 \ge 0, x_2 \ge 0.$

Now consider the functional constraint (1), $x_1 \le 4$. The slack variable of this constraint, x_3, is defined as the difference of right and left side of constraint 1. Therefore, constraint 1 can be rewritten as follows:

$x_1 + x_3 = 4$

The above equation is equal to the limit of 1 provided that $x_3 \ge 0$. Hence constraint 1 is equivalent to the following set of constraints:

$$\left.\begin{array}{c} x_1 + x_3 = 4 \\ x_3 \ge 0 \end{array}\right\} \equiv x_1 \le 4$$

To clarify the subject, we give a simple example. Consider the constraint $x_1 \leq 4$. If $x_1 = 1$ then we have $1 < 4$ and the value of the slack variable is equal to 3, $x_3 = 3$. The relation $x_1 \leq 4$ becomes $x_1 + x_3 = 4$ and in this case $x_1 = 1$ and $x_3 = 3$.

For other inequalities, the model (P1) can be similarly rewritten as follows:

$(P1-1)$ $Max\ Z = 3x_1 + 5x_2$
$s\,t.$

$$
\begin{array}{lllll}
(1) & x_1 & + x_3 & & = 4 \\
(2) & & 2x_2 & + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 & & + x_5 & = 18 \\
\end{array}
$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Model (P1-1) is exactly the same as model (P1), but this new format is much simpler for algebraic operations. In model (P1-1) the number of variables is equal to 5 and the number of constraints is equal to 3, in which we are faced with a set of constriant with two degrees of freedom(number of variables-number of constriants). In this case, you can set the desired value for two additional variables in each step and solve the system equation of three constraint having three variables. In the simplex method, these two additional variables are set equal to zero. Variables that are considered zero are called **Non-basic variables** and others are called **Basic variables**. The solution that all non-basic variables are equal to zero is called the **Basic solution** and the basic solution in which the basic variables are non-negative is called the **Basic feasible solution**.

Notice that the basic variable can be read immediately in the model because each equation has just one basic variable, which has a coefficient of 1, and this basic variable does not appear in any other equation.

It is easier for solving that the objective function rewrite as the functional constraint in the list of constraint with row zero. Therefore, before presenting the simplex method, we rewrite the problem (P1-1) as follows.

$$(P1-2) \ Max \ Z$$
$$s\,t.$$
$$(0)Z \ -3x_1 - 5x_2 \qquad\qquad = 0$$
$$(1) \quad\ x_1 \qquad + x_3 \qquad\quad = 4$$
$$(2) \qquad\quad 2x_2 \qquad + x_4 \qquad = 12$$
$$(3) \quad 3x_1 + 2x_2 \qquad\quad + x_5 = 18$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

In the row zero, Z is the basic variable; because it has a coefficient of 1 in row zero, and it does not appear in any other equation

The primal simplex method

In this section, we will present the formal form of simplex method. In short, at each step the simplex method seeks to find the basic feasible solution provided that this solution is not worse than the previous solution to finally find an optimal basic feasible solution. To convert from one basic feasible solution to another on, it is sufficient to convert one basic variable to a non-basic variable (leaving variable) and a non-basic variable to a basic variable (entering variable). With these changes, the current basic feasible solution moves to the adjacent basic feasible solution. If a basic feasible solution is better than all the adjacent basic solution, that solution is the optimal solution and the algorithm ends at this point. The following figure summarizes the steps of the simplex method in general.
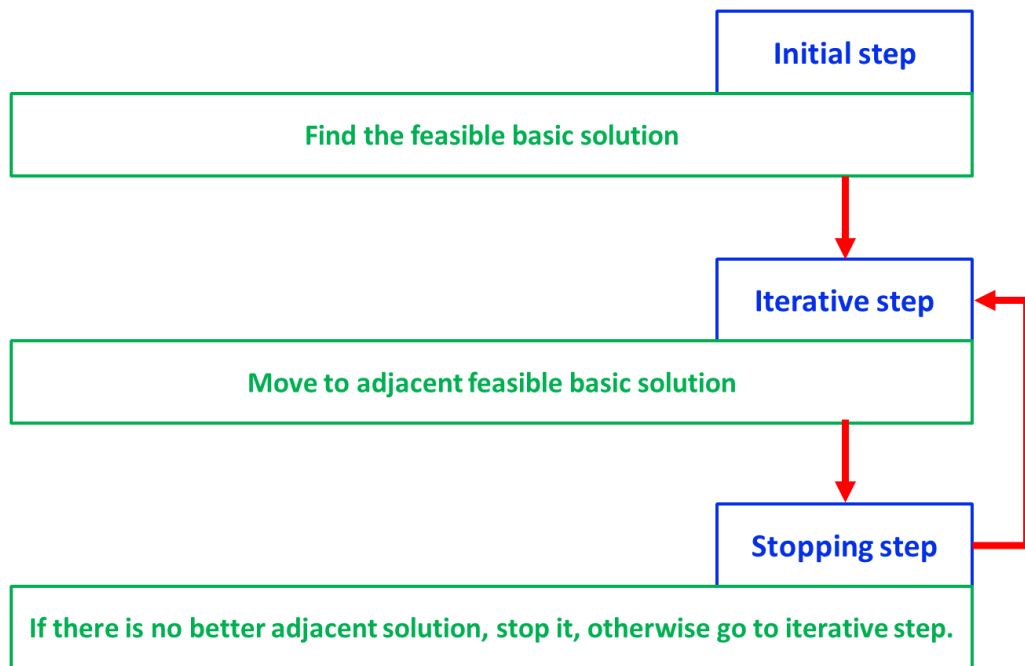
We express each step of the simplex method in the form of an example.

Intial step:

Call the slack variables (x_3, x_4, x_5) as the basic variables and set the (x_1, x_2) variables as non-basic variables equal to zero. For simplicity of calculation, the coefficients of the variables and the value on the right hand side are written in the table named the **simplex tableau**. The example simplex tableau (P1-2) is as follows.

Basic variable	Row	Z	X₁	X₂	X₃	X₄	X₅	Right Hand Side
Z	0	1	-3	-5	0	0	0	0
X₃	1	0	1	0	1	0	0	4
X₄	2	0	0	2	0	1	0	12
X₅	3	0	3	2	0	0	1	18

Because each equation contains a basic variable with a coefficient of +1, so the value of each basic variable will be equal to the value of right hand side

(RHS) of the equation. The basic feasible solution of the above table is equal to $(x_1 = 0, x_2 = 0, x_3 = 4, x_4 = 12, x_5 = 18)$, which is displayed as $(0, 0, 4, 12, 18)$ in the rest.

Stoping step:

If and only if all coefficients of row zero are non-negative (≥ 0), then the current basic feasible solution is the optimal, and then stop. Otherwise, repeat the interative step to find the adjacent basic feasible solution.

Iterative step:

Substep 1: Select the variable that has the largest negative coefficient for non-basic variable in row zero as the **entering variable**. Increasing the value of this non-basic variable leads to the fastest growth rate of the objective function. Draw a rectangle around the column below entering variable and call **pivot column**. In this example, the largest negative coefficient (-5) is for the variable (x_2) and therefore it is selected as the entering variable.

Substep 2: The **leaving basic variable** is determined as follows.

A) Consider the positive coefficients of the pivote column

B) Divide the right hand side into positive coefficients

C) Select the row for which the ratio obtained in part (B) is the smallest.

D) The basic variable of this row is the leaving basic variable.

Substep 3: Draw a rectangle around this row and call **pivot row**. The value in both rectangles (pivote column and pivote row) is called the **pivot number**.

The results of the above operations are given in the table below.

Basic variable (0)	Row (1)	Z (2)	X_1 (3)	X_2 (4)	X_3 (5)	X_4 (6)	X_5 (7)	RHS (8)	Ratio test
Z	0	1	-3	-5	0	0	0	0	
X_3	1	0	1	0	1	0	0	4	Min 6=12/2
X_4	2	0	0	2	0	1	0	12	9=18/2
X_5	3	0	3	2	0	0	1	18	

In the above table, the variable x_2 is the entering variable and the variable x_4 is the leaving variable.

Substep 4: Get the new basic solution with the help of a new simplex tableau. The steps to obtain a new tableau are as follows.

A) In column (0), delete the leaving variable, x_4, and replace it with the entering variable, x_2,.

B) To convert the coefficient of entering variable to **one**, divide the pivote row by the pivote number.

C) In order to remove the e basic variable from the other row, each row (even row zero) except the pivote row should be changed as follows.

C1. Multiply the pivote row by a nonzero constant which lead to become zero the coefficient of pivote column except from pivote number

C2. Add a multiple of pivote row to another row.

After creating a new simplex tableau, we go to the stopping step. If the stopping condition is met, the algorithm will stop; otherwise we will go to iterative step. We will continue this process until the stopping condition is met. The complete table of samples for (P2-1) is as follows.

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side	Ratio test
Z	0	1	-3	-5	0	0	0	0	
X_3	1	0	1	0	1	0	0	4	
X_4	2	0	0	2	0	1	0	12	Min 6=12/2
X_5	3	0	3	2	0	0	1	18	9=18/2
Z	0	1	-3	0	0	2.5	0	30	
X_3	1	0	1	0	1	0	0	4	4=4/1
X_2	2	0	0	1	0	0.5	0	6	
X_5	3	0	3	0	0	-1	1	6	Min 2=6/3
Z	0	1	0	0	0	1.5	1	36	
X_3	1	0	0	0	1	0.333	-0.333	2	
X_2	2	0	0	1	0	0.5	0	6	
X_1	3	0	1	0	0	-0.333	0.333	2	

In the table above, the basic solution $(2,6,2,0,0)$ with $Z = 36$ satisfies the stopping condition, and therefore this solution is optimal.

Note: In each iteration of the simplex algorithm, one of the corner points of the feasible region is tested. Due to the small size of the model, the movement on the corner points of the feasible region can be seen in the graphical display of the model (P1) as follows.
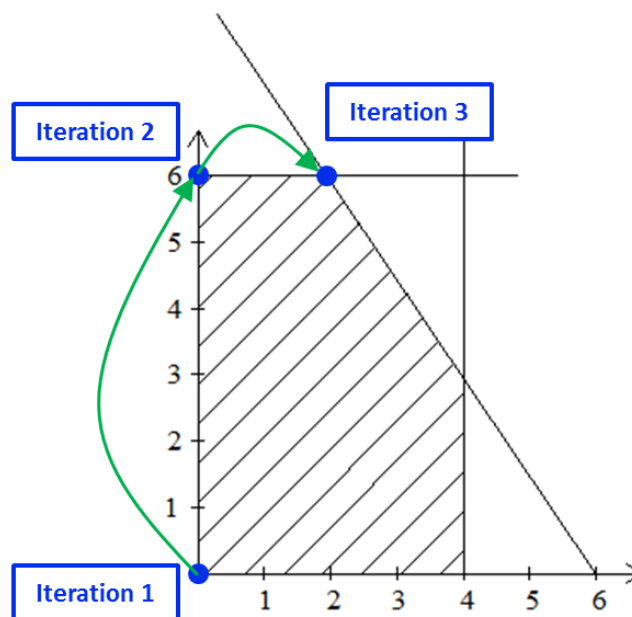
If we had not chosen the larger absolute negative coefficient(it means x_1 is as a entering variable), we would moved counterclockwise in the feasible region, and another basic feasible solution having been considered.

Shadow price

The simplex method produces other valuable information in addition to the optimal solution. The shadow value of the source i (denoted by y_i^*) measures the marginal value of source i, which indicates the rate of increase of Z due to a slight increase in the right hand side of source i (b_i). Note that the rate of increase must be small enough that the current set of variables remains optimal because as soon as the set of basic variables changes, the shadow price also changes. The coefficient of i-th slack variable, which is related to i-th constraint, in row zero of final simplex tabluea determines the shadow price of i-th constraint.

For example in model (P1), the shadow price of constraint 1 is equal to the coefficient of x_3 in row zero (equal to zero), the shadow price of constraint 2 is equal to the coefficient of x_4 in row zero (equal to 1.5) and the shadow price of constraint 3 is equal to the coefficient of x_5 in row zero (equal to 1).

Note: In other words, the shadow price of source 1 is the maximum price that it is cost-effective to pay to increase one unit of this source.

Special cases in simplex method

Picking the entering variable

Suppose two or more non-basic variables have the largest negative coefficient. Here, pick the one with the largerst index. For example, consider (P1) model with different objective function, Z=3x_1+3x_2. Both x_1 and x_2 had a coefficient of 3, and 3; we pick x_2. In this problem, if x_1 is selected, we will

reach the optimal answer after three iterations, while if we choose x_2 , we will reach the optimal solution after two iterations.

Degeneracy

A linear model is degenerate if in a basic feasible solution, one of the basic variables takes on a zero value. Degeneracy is a problem in practice, because it makes cycling in the basic solution and then makes the simplex algorithm slower.

Note: A simple rule to get out of the state of decay is to use Bland's rule. In this rule, a variable is selected as an input to the base that has the lowest index and the lowest coefficient of the objective function. In the case that there is more than one choice for the output variable from the base, the variable that is closest to the target function line or the zero line is selected.

Unbounded Z

Consider a situation where none of the basic variables have leaving conditions, and the value of the basic entering variables can increase infinitely without any of the other basic variables being negative. This condition occurs when coefficients of the pivote column in the simplex table are all negative or zero (non-positive). Consider the following model to clarify the issue.

$$Max \ Z = 3x_1 + 5x_2$$
$$s\,t.$$
$$(1) \quad x_1 \qquad \leq 4$$
$$(2) \qquad -x_2 \leq 12$$
$$(3) \quad 3x_1 - 2x_2 \leq 18$$
$$x_1 \geq 0, x_2 \geq 0.$$

The graphical display of the above model is as follows. Clearly, the value of the objective function increases infinitly and the form of a simplex table is given below.
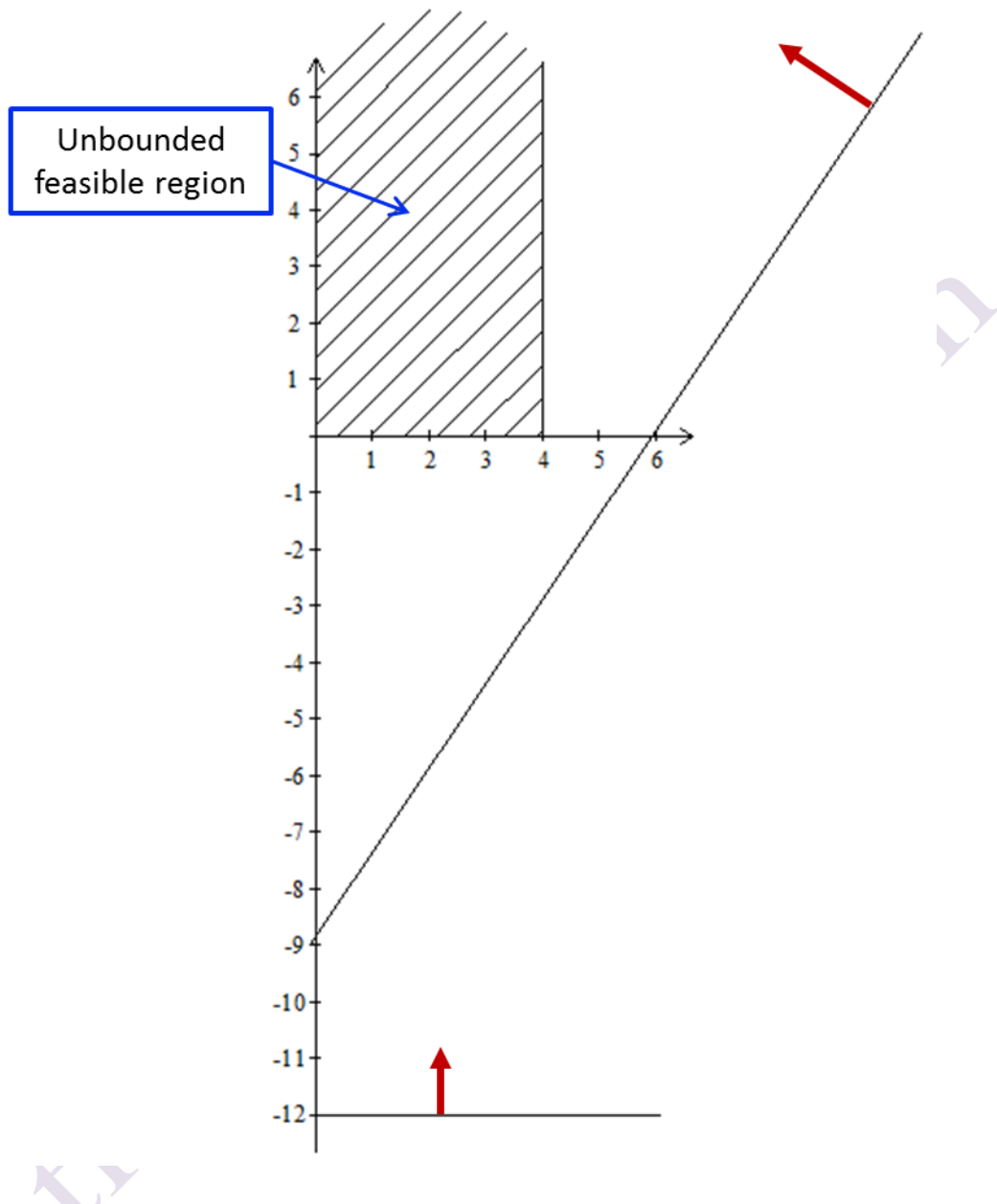
Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side
Z	0	1	-3	-5	0	0	0	0
X_3	1	0	1	0	1	0	0	4
X_4	2	0	0	-2	0	1	0	12
X_5	3	0	3	-2	0	0	1	18

In the table above, we have $x_4 = 12 + 2x_2$ and $x_5 = 18 + 2x_2 - 3x_1$ where x_2 can be increased to infinity. Therefore, the value of the objective function increases to infinity and the constraints do not prevent the unboundness of

objective function. If this happened, it could have been a miscalculation or missing some constraints.

Multiple optimal solutions

Whenever a problem has more than one optimal basic feasible solution, at least one of the nonbasic variables has a coefficient of zero in the final row zero, so increasing any such variable will not change the value of Z. Therefore, these other optimal basic feasible solutions can be identified (if desired) by performing additional iterations of the simplex method, each time choosing a nonbasic variable with a zero coefficient as the entering basic variable.

Consider the following model

$$Max \ Z = 3x_1 + 2x_2$$

$s\,t.$

(1) $\quad x_1 \qquad \leq 4$

(2) $\qquad 2x_2 \leq 12$

(3) $\quad 3x_1 + 2x_2 \leq 18$

$\qquad x_1 \geq 0, x_2 \geq 0.$

The graphical display of the above model is given below. In this figure, each point of the line connecting (2,6) and (4,3) is the optimal solution. Knowing the existence of multiple optimal solutions to the problem is often very important in applying the results in practice. A mathematical model cannot take into account all the economic factors in the real world. Therefore, you can choose the best solution of optimal solutions with the help of an expert.

$$Z = 3x_1 + 2x_2$$
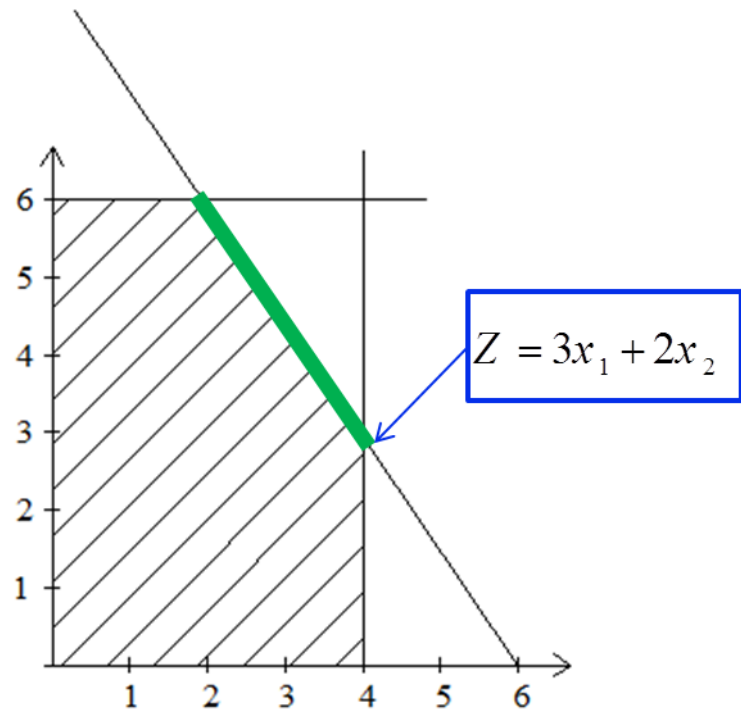
Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	Right Hand Side	
Z	0	1	-3	-2	0	0	0	0	Non optimal
X_3	1	0	1	0	1	0	0	4	Min 4=4/1
X_4	2	0	0	2	0	1	0	12	
X_5	3	0	3	2	0	0	1	18	6=18/3
Z	0	1	0	-2	3	0	0	12	Non optimal
X_1	1	0	1	0	1	0	0	4	
X_4	2	0	0	2	0	1	0	12	6=12/2
X_5	3	0	0	2	-3	0	1	6	Min 3=6/2
Z	0	1	0	0	0	0	1	18	**Optimal**
X_1	1	0	1	0	1	0	0	4	4=4/1
X_4	2	0	0	0	3	1	-1	6	Min 2=6/3
X_2	3	0	0	1	-1.5	0	0.5	3	
Z	0	1	0	0	0	0	1	18	**Optimal**
X_1	1	0	1	0	0	-0.333	0.333	2	
X_3	2	0	0	0	1	0.333	-0.333	2	
X_2	3	0	0	1	0	0.5	0	6	

In the table above, the two basic solution $(4,3,0,6,0)$ and $(2,6,2,0,0)$ are optimal and the optimal value is $Z = 18$ in both.

Equality constriants

Any equality constraint

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = b_i$$

actually is equivalent to a pair of inequality constraints:

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \geq b_i$$

$$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \leq b_i$$

However, rather than making this substitution and thereby increasing the number of constraints, it is more convenient to use the artificial-variable technique. We shall illustrate this technique with the following example.

The only resulting change in the linear programming model is that the third constraint, $3x_1 + 2x_2 \leq 18$, instead becomes an equality constraint. The new model as follows:

$$Max\ Z$$
$$s.t.$$
$$(0)\ Z - 3x_1 - 5x_2 \qquad\qquad = 0$$
$$(1) \qquad x_1 \qquad\quad + x_3 \qquad = 4$$
$$(2) \qquad\qquad 2x_2 \qquad\ + x_4 = 12$$
$$(3) \qquad 3x_1 + 2x_2 \qquad\qquad = 18$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The below figure shows the graphical display of the above model:
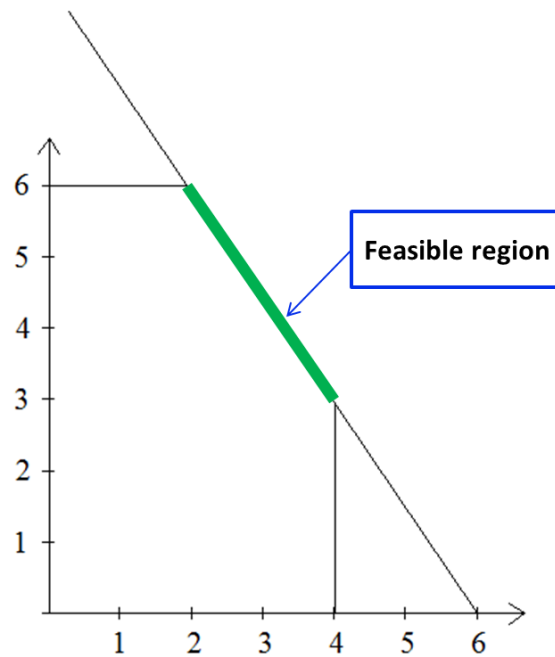
Unfortunately, these equations do not have an obvious initial basic feasible solution because there is no longer a slack variable to use as the initial basic variable for constriant (3). It is necessary to find an initial basic solution solution to start the simplex method.

This difficulty can be circumvented in the following way.

Obtaining an Initial BF Solution.

The procedure is to construct an artificial problem that has the same optimal solution as the real problem by making two modifications of the real problem.
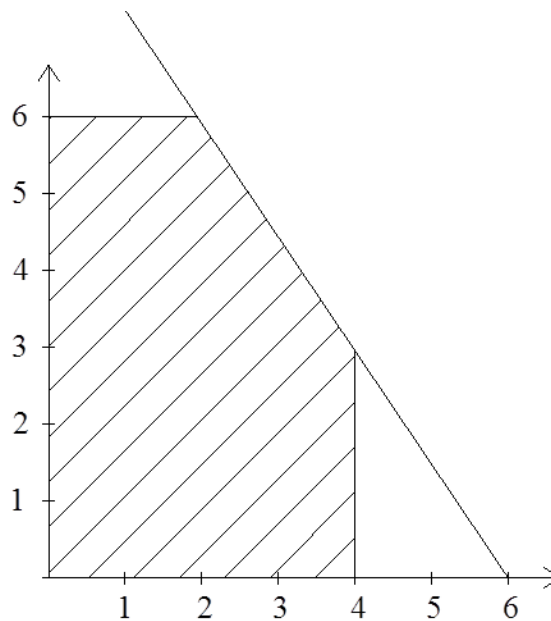
1. Apply the **artificial variable** by introducing a nonnegative artificial variable (call it $\bar{x}_5$) into constraint (3), then we have:

$$3x_1 + 2x_2 + \bar{x}_5 = 18$$

And then the model would be:

$Max\ Z$

$s\,t.$

$(0)\,Z - 3x_1 - 5x_2 \qquad\qquad\qquad = 0$

$(1)\qquad x_1 \qquad + x_3 \qquad\quad = 4$

$(2)\qquad\qquad 2x_2 \qquad + x_4 \quad = 12$

$(3)\qquad 3x_1 + 2x_2 \qquad\quad + \bar{x}_5 = 18$

$\qquad\quad x_1, x_2, x_3, x_4, \bar{x}_5 \geq 0$

Adding an artificial variable, $\bar{x}_5$, enlarges the possible area similar to the figure below.

2. Assign the penalty to having $\bar{x}_5 = 0$ by changing the objective function $Z - 3x_1 - 5x_2 + M\bar{x}_5 = 0$, where M symbolically represents a huge positive number. (This method of forcing $\bar{x}_5 = 0$ in the optimal solution is called the **Big M method**.)

Converting Equation (0) to Proper Form.

The system of constraints is not yet in proper form because a basic variable $\bar{x}_5$ has a nonzero coefficient in Eq. (0). Recall that all basic variables must be algebraically eliminated from Eq. (0) before the simplex method can either apply the optimality test or find the entering basic variable. To

algebraically eliminate $\bar{x}_5$ from row zero, we need to subtract from row zero the product, M times row 3, as below.

Row 0	-3	-5	0	0	M	0	Ⓐ
Row 3	3	2	0	0	1	18	Ⓑ

Row 0	(-3M-3)	(-2M-5)	0	0	1	18	Ⓐ+(-M)Ⓑ

Now find the optimal solution for the real problem by applying the simplex method to the artificial problem, starting with the following initial BF solution:

Basic variable	Row	Z	X_1	X_2	X_3	X_4	$\bar{X}_5$	Right Hand Side	
Z	0	1	-3M-3	-2M-5	0	0	0	-18M	Non optimal
X_3	1	0	1	0	1	0	0	4	Min 4=4/1
X_4	2	0	0	2	0	1	0	12	
$\bar{X}_5$	3	0	3	2	0	0	1	18	6=18/3
Z	0	1	0	-2M-5	3M+3	0	0	-6M+12	Non optimal
X_1	1	0	1	0	1	0	0	4	
X_4	2	0	0	2	0	1	0	12	6=12/2
$\bar{X}_5$	3	0	0	2	-3	0	1	6	Min 3=6/2
Z	0	1	0	0	-4.5	0	M+2.5	27	Non optimal
X_1	1	0	1	0	1	0	0	4	4=4/1
X_4	2	0	0	0	3	1	-1	6	Min 2=6/3
X_2	3	0	0	1	-1.5	0	0.5	3	
Z	0	1	0	0	0	1.5	M+1	36	Optimal
X_1	1	0	1	0	0	-0.333	0.333	2	
X_3	2	0	0	0	1	0.333	-0.333	2	
X_2	3	0	0	1	0	0.5	0	6	

In the above table, the optimal solution is (2,6,2,0,0).

Two-phase method

As mentioned, the big M method is used to solve the linear programming model with artificial variables. But because calculations with big M are very difficult in large models, it can lead to computational error. In this cas, **Two-phase method** is recommended. In this method, phase 1 seeks to minimize the sum of artificial variables. If the model succeeds in making the sum of the artificial variables equal to zero, the solution is used as the initial solution of the model in phase 2. In the following example, two-phase method is applied..

Example: Using the two-phase method, work through the simplex method step by step to solve the problem.

$$Max\ Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$$
$$s\,t.$$
$$(1)\quad 2x_1 + 3x_2 + 4x_3 + 2x_4 = 300$$
$$(2)\quad 8x_1 + x_2 + x_3 + 5x_4 = 300$$
$$x_i \geq 0 \quad i = 1,...,4.$$

Solution:

Write the model in the standard form:

$$Max\ Z = 4x_1 + 2x_2 + 3x_3 + 5x_4$$
$$s\,t.$$
$$(1)\quad 2x_1 + 3x_2 + 4x_3 + 2x_4 + x_5 = 300$$
$$(2)\quad 8x_1 + x_2 + x_3 + 5x_4 + x_6 = 300$$
$$x_i \geq 0 \quad i = 1,...,6.$$

In the two-phase method, we first solve a model that seeks to minimize the objective function $w = x_5 + x_6$ with the limitations of the original model. In fact, we are looking to optimize the following model.

$$Min\, W = \qquad\qquad x_5 + x_6$$

$s\,t.$

$\quad$ (1) $\quad 2x_1 + 3x_2 + 4x_3 + 2x_4 + x_5 \qquad = 300$

$\quad$ (2) $\quad 8x_1 + x_2 + x_3 + 5x_4 \qquad + x_6 = 300$

$\qquad\qquad x_i \geq 0 \quad i = 1,...,6.$

The solution of the above model is done simultaneously with the orginal model, in which the objective function W and Z are entered in the simplex table.

Basic variable	Row	Z	X₁	X₂	X₃	X₄	X₅	X₆	X₇	RHS	
W		0	-1	0	0	0	0	1	1	0	
Z	0	1	0	4	2	3	5	0	0	0	
X₅	1	0	0	2	3	4	2	1	0	300	
X₆	2	0	0	8	1	1	5	0	1	300	
W		0	-1	-10	-4	-5	-7	0	0	-600	Non optimal – Start phase 1
Z	0	1	0	-4	-2	-3	-5	0	0	0	
X₅	1	0	0	2	3	4	2	1	0	300	300/2=150
X₆	2	0	0	8	1	1	5	0	1	300	**Min 300/8=75**
W		0	-1	0	-2.75	-3.75	-0.75	0	1.25	-225	Non optimal – Continue phase 1
Z	0	1	0	0	-1.5	-2.5	-2.5	0	0.5	150	
X₅	1	0	0	1	2.75	3.75	0.75	1	-0.25	225	**Min 60=225/3.75**
X₁	2	0	0	0	0.125	0.125	0.625	0	0.125	375	3000=375/0.125
W		0	-1	0	0	0	0	1	1	0	**Optimal – End phase 1**
Z	0	1	0	0	0.333	0	-2	0.666	0.333	300	
X₃	1	0	0	0	0.733	1	0.2	0.266	-0.067	60	
X₁	2	0	0	1	0.033	0	0.6	-0.033	0.133	30	
Z	0	1	0	0	0.333	0	-2			300	Non optimal – Start phase 2
X₃	1	0	0	0	0.733	1	0.2			60	300=60/0.2
X₁	2	0	0	1	0.033	0	0.6			30	**Min 50=30/0.6**
Z	0	1	0	3.33	0.44	0	0			400	**Optimal – End phase 2**
X₃	1	0	0	-0.333	0.722	1	0			50	
X₄	2	0	0	1.666	1.666	0	1			50	

The optimal solution of original model is $(0,0,50,50)$ and $Z^* = 400$.

Revised simplex method

Another practical method for solving linear programming problems is the revised simplex method. Although the simplex method is suitable for performing manual calculations, it does not have the necessary efficiency to

solve large problems by computer. The reason can be found in the storage of information that may never be used in simplex iterations. For example, some variables never meet the necessary conditions to be selected as input variables, as a result, all calculations related to the coefficients of these variables in the objective function and constraints will remain unused. In fact, the column corresponding to that variable is calculated, but it is not actually used.

The Revised Simplex method solves the problem by using all the principles and steps of the original Simplex method, without performing redundant operations and storing useless information, which require a lot of memory to keep in the computer. The use of the matrix is an inevitable necessity in applying the revised simplex method, and this makes the definition of the matrix of the linear programming problem necessary.

Matrix representation of the linear programming

In general, the standard linear programming problem is represented as follows:

$$Max\ Z = cx$$
$$Ax \leq \bar{b}$$
$$x \geq 0$$

Where

$$x = \left[x_1, x_2, ..., x_n\right]^T \quad c = \left[c_1, c_2, ..., c_n\right]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \quad \bar{b} = \left[\bar{b}_1, \bar{b}_2, ..., \bar{b}_m\right]^T$$

Example: Display the following model as a matrix.

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 3x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

In the above model:

$$x = [x_1, x_2, x_3]^T \quad c = [4,3,6]$$
$$A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix} \quad \bar{b} = [30, 40]^T$$
$$\rightarrow Max\ Z = [4,3,6]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In case of adding slack variables, the problem becomes as follows:

$$\begin{matrix} & Max\ Z = cx \\ Max\ Z = cx & \\ Ax + Is = \bar{b} \quad \rightarrow & [A,I]\begin{bmatrix} x \\ s \end{bmatrix} = \bar{b} \\ x, s \ge 0 & \\ & x, s \ge 0 \end{matrix}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

where I is called the m-by-m unit matrix and

$$s = [s_1, s_2, ..., s_m]^T$$

By adding the slack variable s, the model is displayed as a matrix representation:

$$Max\ Z = [4,3,6,0,0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 1 & 3 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \end{bmatrix}}_{[A,I]} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

The number of basic variables is equal to the number of constraints and is equal to m. The matrix [A,I] has m rows (m number of constraints other than non-negativity constraints) and m+n columns, which include m basic variables and n non-basic variables. In the above example, m is equal to 2 and n is equal to 3.

Matrix [A,I] is divided into two matrices, one of which consists of the coefficients of basic variables and the other of the coefficients of non-basic variables. The matrix of coefficients of basic variables is represented by B and the matrix of coefficients of non-basic variables is represented by N. In the same way, all the variables of the problem, $x_j, s_i\ (j = 1,...,m; i = 1,...,n)$, are also divided into two groups of basic variables (x_B) and non-basic variables (x_N).

Note: In the first step of the basic simplex algorithm, basic variables are of type s_i and non-basic variables are of type x_j. In the next steps, this situation will change and the basic variables will be a combination of s_i and x_j. In the following, we will explain this situation with an example.

Example: Solve the following model using the revised simplex method.

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + x_2 + 3x_3 \leq 30$$
$$2x_1 + 2x_2 + 3x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$

Solution:

The initial simplex table for the above problem is as follows:

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-4	-3	-6	0	0	0
s_1	1	0	3	1	3	1	0	30
s_2	2	0	2	2	3	0	1	40

$\overline{a}_1 \nearrow$ $B \nearrow$ $\overline{b} \nearrow$

In the above table, basic variables or x_B are slack variables (s_1,s_2); and non-basic variables or x_N are decision variables (x_1,x_2,x_3). B is the matrix of coefficients of basic variables and N is the matrix of coefficients of non-basic variables in the constraints.

$$x_B = [s_1, s_2], x_N = [x_1, x_2, x_3]$$

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, N = \begin{bmatrix} 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix}$$

The table of the first iteration of the simplex method is as follows.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	2	-1	0	2	0	60
x_3	1	0	1	$\frac{1}{3}$	1	$\frac{1}{3}$	0	10
s_2	2	0	-1	1	0	-1	1	10

$B^{-1} \nearrow$

In the above table

$$x_B = [x_3, s_2], x_N = [x_1, x_2, s_1]$$

$$B = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}, N = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

The matrix B is the coefficients of the basic variables in the initial table, and B^{-1} is the inverse matrix of B. The coefficient of x_3 is 3 in the first constraint

and 3 in the second constraint, so the first column of matrix B is created in this way.

$$Max \ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + \ x_2 + \boxed{3}x_3 \le 30$$
$$2x_1 + 2x_2 + \boxed{3}x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

The final simplex table is as follows.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	1	0	0	1	1	70
x_3	1	0	$\frac{4}{3}$	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
x_2	2	0	-1	1	0	-1	1	10

$$B^{-1} \nearrow$$

Where

$$x_B = [x_3, x_2], x_N = [x_1, s_1, s_2]$$
$$B = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix}, N = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The matrix B is the coefficients of the basic variables in the initial table, and B^{-1} is the inverse matrix of B. The coefficient of x_3 is 3 in the first constraint and 3 in the second constraint, so the first column of matrix B is created in this way. The x_2 coefficient is 1 in the first constraint and 2 in the second constraint, so the second column of matrix B is created in this way.

With the explanations given above, the matrix representation of the linear programming model can be expressed. The constraints of the linear programming model are written as follows.

$$[A, I]\begin{bmatrix} x \\ s \end{bmatrix} = \bar{b}$$

The variables are divided into two parts according to decision and slack variables. It can also be divided into two parts according to basic and non-basic variables and written as follows:

$$[B,N]\begin{bmatrix} x_B \\ x_N \end{bmatrix} = \bar{b} \rightarrow Bx_B + Nx_N = \bar{b}$$

If we multiply both sides of the equation by B^{-1}, we will have:

$$B^{-1}Bx_B + B^{-1}Nx_N = B^{-1}\bar{b}$$
$$x_B = B^{-1}\bar{b} - B^{-1}Nx_N$$

Since x_N represent non-basic variables and have zero value, we have as a result:

$$x_B = B^{-1}\bar{b}$$

If we have B^{-1} for each table, the above relation gives the value of the basic variables of that table or the numbers on the right side of it.

The objective function will be as follows by separating the variables into basic and non-basic variables:

$$Z = c_B x_B + c_N x_N$$
$$Z = c_B\left(B^{-1}\bar{b} - B^{-1}Nx_N\right) + c_N x_N$$
$$Z = c_B B^{-1}\bar{b} - c_B B^{-1}Nx_N + c_N x_N$$
$$Z = c_B B^{-1}\bar{b} - \left(c_B B^{-1}N - c_N\right)x_N$$

Since x_N represent non-basic variables and have zero value, we have as a result:

$$Z = c_B B^{-1}\bar{b}$$

The coefficient of the non-basic variables in the objective function, that is, the coefficient x_N, which is denoted by the symbol $z_j - c_j$, will be as follows according to the above relationships:

$$z_j - c_j = c_B B^{-1}N - c_N$$

Coefficients of non-essential variables (x_N) in the constraints, according to the following equation

$$x_B = B^{-1}\overline{b} - B^{-1}Nx_N$$

Equal to

$$B^{-1}N$$

The N matrix is shown as a column vector of non-basic variables, each column corresponding to one variable.

$$N = [\overline{a}_1, \overline{a}_2, ..., \overline{a}_n]$$

where each column of N shows the variable coefficients x_j in the constraints. In this way, the following relationship is used to calculate the pivote column numbers:

$$a_j = B^{-1}\overline{a}_j$$

To learn more about how to apply the above relations, we give an example.

Example:

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 3x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

Suppose we want to obtain the numbers of the simplex table, where the variables x_3 and x_2 are the basic variables. Therefore, we have:

$$c_B = \begin{bmatrix} \overset{x_3\ x_2}{6, 3} \end{bmatrix}, c_N = \begin{bmatrix} \overset{x_1\ s_1\ s_2}{4, 0, 0} \end{bmatrix}, B = \begin{bmatrix} \overset{x_3\quad x_2}{3} & 1 \\ 3 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix}$$

$$x_N = [x_1, s_1, s_2], x_B = [x_3, x_2]$$

To calculate the coefficient of non-basic variables in the objective function of the simplex table, where x_3 and x_2 are basic variables, it is done as follows:

$$z_j - c_j = c_B B^{-1} N - c_N = \begin{bmatrix} x_3 & x_2 \\ 6, 3 \end{bmatrix} \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & s_1 & s_2 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_1 & s_1 & s_2 \\ 4, 0, 0 \end{bmatrix} = \begin{bmatrix} x_1 & s_1 & s_2 \\ 1, 1, 1 \end{bmatrix}$$

In this simplex table, the value of x_B and Z is calculated as follows.

$$x_B = B^{-1}\bar{b} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} \dfrac{20}{3} \\ 10 \end{bmatrix}$$

$$Z = c_B B^{-1}\bar{b} = \begin{bmatrix} 6, 3 \end{bmatrix} \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = 70$$

The calculation of the x_1 variable column is done as follows:

$$a_1 = B^{-1}\bar{a}_1$$

$$a_1 = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \dfrac{4}{3} \\ 1 \end{bmatrix}$$

What has been said so far has covered all the necessary prerequisites for presenting the revised simplex algorithm. Therefore, we will continue to introduce this algorithm.

Revised simplex algorithm

Step 1: Determine the input variable. At each iteration, find the non-prime coefficients in the objective function using the $c_B B^{-1} N - c_N$. If these values are all non-negative, you have reached the **optimal solution**. Calculate the optimal solution using the following relations.

$$x_B = B^{-1}\bar{b}$$
$$Z = c_B B^{-1}\bar{b}$$

Otherwise, select the variable with the most negative calculated coefficient. This variable is the **input variable**.

Step 2: Determine the output variable. Selecting the **output variable** requires having the coefficients of the input variable in the constraints and numbers on the right hand side. If the variable x_j is the input variable, its coefficients in the constraints are obtained from the following relationship.

$$a_j = B^{-1}\overline{a}_j$$

The value of the current basic variables which is equal to numbers on the right hand side is:

$$x_B = B^{-1}\overline{b}$$

The basic output variable is obtained as follows.

$$\underset{j}{Min}\left\{\frac{x_B}{a_j}\right\}$$

Step 3: Calculate the new B^{-1} and determine **the new basic variable** and go to step 1.

Example: Consider the following model. Obtain the optimal solution using the revised simplex algorithm.

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 3x_3 \le 40$$
$$x_1, x_2, x_3 \ge 0$$

Solution:

We rewrite the model as follows.

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$3x_1 + x_2 + 3x_3 + s_1 \quad\quad = 30$$
$$2x_1 + 2x_2 + 3x_3 \quad\quad + s_2 = 40$$
$$x_1, x_2, x_3, s_1, s_2 \ge 0$$

Iteration 1

Step 1. Determine the input variable. The basic variable of iteration 1, s_1 and s_2 and their coefficient in the objective function is zero ($c_B=(0,0)$). The coefficients of non-basic variables ($x_N=(x_1,x_2,x_3)^T$) are calculated as follows:

$$z_j - c_j = c_B B^{-1} N - c_N = \begin{bmatrix} s_1 & s_2 \\ 0,0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ 3 & 1 & 3 \\ 2 & 2 & 3 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 & x_3 \\ 4, & 3, & 6 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ -4, & -3, & -6 \end{bmatrix}$$

Because the coefficient of x_3 is equal to -6 and the most negative number, this is the input variable shown in the table below.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-4	-3	-6	0	0	0

$$-6 = \min\{-4, -3, -6\}$$

Step 2. Determine the output variable. The x_3 coefficients in the constraints are calculated as follows:

$$a_3 = B^{-1} \overline{a}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

$$\rightarrow Min\left\{ \frac{30}{3}, \frac{40}{3} \right\} = 10$$

Because the smallest number of the result corresponds to the basic variable s_1, this variable is selected as the output variable. The calculations of steps 1 and 2 are summarized as follows:

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-4	-3	-6	0	0	0
s_1	1				3			30
s_2	2				3			40

Step 3. Calculate the new B and x_B. Since variable x_3 is input and s_1 is output, the basic variables of the new iteration are:

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$$x_B = [x_3, s_2]^T, c_B = [6,0] \rightarrow B = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix}$$

Iteration 2

Step 1. Determine the input variable. The coefficients of non-basic variables, $x_N = [x_1, x_2, s_1]$, are calculated as follows.

$$z_j - c_j = c_B B^{-1} N - c_N = \overset{x_3 \; s_2}{[6,0]} \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix} \overset{x_1 \; x_2 \; s_1}{\begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}} - \overset{x_1 \; x_2 \; s_1}{[4, 3, 0]} = \overset{x_1 \; x_2 \; s_1}{[2, \underset{\Uparrow}{-1}, 2]}$$

x_2 becomes the input variable.

Step 2. Determine the output variable.

$$a_2 = B^{-1}\overline{a}_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_3 \\ s_2 \end{bmatrix} = B^{-1}\overline{b} = \begin{bmatrix} \frac{1}{3} & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\rightarrow Min \left\{ \overset{x_3 \quad s_2}{\frac{10}{\frac{1}{3}}, \frac{10}{\underset{\Uparrow}{1}}} \right\} = 10$$

s_2 is the output variable.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1		-1				
x_3	1	0		$\frac{1}{3}$				10
s_2	2	0		1				10

Step 3. Since x_2 is the input variable and s_2 is the output variable, the basic variables of the next iteration are:

$$x_B = \begin{bmatrix} x_3, x_2 \end{bmatrix}^T, c_B = \begin{bmatrix} 6,3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 3 & 2 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix}$$

Iteration 3

Step 1. Determine the input variable. The coefficients of non-basic variables ($x_N = [x_1, s_1, s_2]^T$) are calculated as follows:

$$z_j - c_j = c_B B^{-1} N - c_N = \overset{x_3 \ x_2}{\begin{bmatrix} 6, 3 \end{bmatrix}} \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -1 & 1 \end{bmatrix} \overset{x_1 \ s_1 \ s_2}{\begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}} - \overset{x_1 \ s_1 \ s_2}{\begin{bmatrix} 4, 0, 0 \end{bmatrix}} = \overset{x_1 \ s_1 \ s_2}{\begin{bmatrix} 1, 1, 1 \end{bmatrix}}$$

Because all the obtained numbers are non-negative, you have reached the optimal solution.

Example: Solve the following problem by the revised simplex method.

$$\begin{aligned} Max \ Z &= 3x_1 + 2x_2 \\ 2x_1 &+ x_2 \le 4 \\ x_1 &+ 2x_2 \ge 6 \\ x_1 &, x_2 \ge 0 \end{aligned}$$

Solution:

The s_1 variable is considered as the basic variable of the first constraint, the s_2 variable is considered to equalize the second constraint, and the R_2 variable is considered as a slack and artificial variable, respectively, for the purpose of the second constraint basic variable.

$$\begin{aligned} Max \ Z &= 3x_1 + 2x_2 && - MR_2 \\ 2x_1 &+ x_2 + s_1 && = 4 \\ x_1 &+ 2x_2 && - s_2 + R_2 = 6 \\ x_1 &, x_2, s_1, s_2, R_2 \ge 0 \end{aligned}$$

Thus we have:

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$$x_B = [s_1, R_2]^T \quad x_N = [x_1, x_2, s_2]^T \quad c_B = [0, -M] \quad c_N = [3, 2, 0]$$

$$B = B^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Iteration 1:

Step 1. We define the input variable.

$$z_j - c_j = c_B B^{-1} N - c_N = \overset{s_1 \quad R_2}{[0, -M]} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{x_1 \quad x_2 \quad s_2}{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}} - \overset{x_1 \, x_2 \, s_2}{[3, 2, 0]}$$

$$= \left[\overset{x_1}{\overbrace{-M - 3}}, \underset{\Uparrow}{\overset{x_2}{\underbrace{-2M - 2}}}, \overset{s_2}{M} \right]$$

Considering that the value of -2M-2 has the most negative value, therefore, the variable x_2 is introduced as an input variable to the basic variable.

Step 2. Select the output variable from the base variables. For this, we must first obtain the value of variable coefficients x_2 in all constraints.

$$a_2 = B^{-1} \bar{a}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$x_B = \begin{bmatrix} s_1 \\ R_2 \end{bmatrix} = B^{-1} \bar{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\rightarrow Min \left\{ \overset{s_1 \quad R_2}{\frac{4}{1}, \underset{\Uparrow}{\frac{6}{2}}} \right\} = 3$$

The variable R_2 is removed from the base.

Step 3. We calculate the new B^{-1}.

$$x_B = [s_1, x_2]^T \quad x_N = [x_1, R_2, s_2]^T \quad c_B = [0, 2] \quad c_N = [3, -M, 0]$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ 0 & \dfrac{1}{2} \end{bmatrix}$$

Iteration 2

Step 1. Determine the input variable.

$$z_j - c_j = c_B B^{-1} N - c_N = \overset{s_1 \; x_2}{[0, 2]} \begin{bmatrix} 1 & -\dfrac{1}{2} \\ 0 & \dfrac{1}{2} \end{bmatrix} \overset{x_1 \quad R_2 \quad s_2}{\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}} - \overset{x_1 \quad R_2 \quad s_2}{[3, -M, 0]}$$

$$= \begin{bmatrix} x_1 & R_2 & s_2 \\ -2, M+1, -1 \\ \Uparrow \end{bmatrix}$$

x_1 becomes the input variable to the base.

Step 2. Determine the output variable.

$$a_1 = B^{-1} \bar{a}_1 = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ 0 & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{2} \\ \dfrac{1}{2} \end{bmatrix}$$

$$x_B = \begin{bmatrix} s_1 \\ x_2 \end{bmatrix} = B^{-1} \bar{b} = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ 0 & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\rightarrow Min \left\{ \overset{s_1}{\dfrac{1}{\dfrac{3}{2}}}, \overset{x_2}{\dfrac{3}{\dfrac{1}{2}}} \right\} = \dfrac{2}{3}$$
$$\qquad\qquad \Uparrow$$

The variable s_1 becomes the output variable of the base.

Step 3. We calculate the new B^{-1}.

$$x_B = [x_1, x_2]^T \quad x_N = [s_1, R_2, s_2]^T \quad c_B = [3, 2] \quad c_N = [0, -M, 0]$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix}$$

Iteration 3

Step 1. Determine the input variable.

$$z_j - c_j = c_B B^{-1} N - c_N = \overset{x_1 \; x_2}{\left[3, 2\right]} \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\[2mm] -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix} \overset{s_1 \quad R_2 \quad s_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}} - \overset{s_1 \quad R_2 \quad s_2}{\left[0, -M, 0\right]}$$

$$= \begin{bmatrix} \overset{s_1}{\dfrac{4}{3}}, \overset{R_2}{M + \dfrac{1}{3}}, \underset{\Uparrow}{\overset{s_2}{\dfrac{-1}{3}}} \end{bmatrix}$$

The variable s_2 enters the base.

Step 2. Determine the output variable.

$$a_{s_2} = B^{-1} \bar{a}_{s_2} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\[2mm] -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3} \\[2mm] -\dfrac{2}{3} \end{bmatrix}$$

$$x_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B^{-1} \bar{b} = \begin{bmatrix} \dfrac{2}{3} & -\dfrac{1}{3} \\[2mm] -\dfrac{1}{3} & \dfrac{2}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} \dfrac{2}{3} \\[2mm] \dfrac{8}{3} \end{bmatrix}$$

$$\rightarrow Min \left\{ \underset{\Uparrow}{\overset{x_1}{\dfrac{\frac{2}{3}}{\frac{1}{3}}}}, \overset{x_2}{-} \right\} = 2$$

The ratio test is performed only on the positive numbers of the column corresponding to the input variable. Therefore, the x_1 variable is removed from the base.

Step 3. Calculate the new B^{-1}.

$$x_B = [s_2, x_2]^T \quad x_N = [s_1, R_2, x_1]^T \quad c_B = [0, 2] \quad c_N = [0, -M, 3]$$

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Iteration 4

Step 1. Determine the input variable.

$$z_j - c_j = c_B B^{-1} N - c_N = \overset{s_2 \ x_2}{\left[0, 2\right]} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \overset{s_1 \ R_2 \ x_1}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}} - \overset{s_1 \ R_2 \ x_1}{\left[0, -M, 3\right]}$$

$$= \overset{s_1 \ R_2 \ x_1}{\left[2, M, 1\right]}$$

Because there are no negative values in the above values, we reached the optimal solution. the optimal solution is as follows.

$$Z^* = c_B B^{-1} \overline{b} = \overset{s_2 \ x_2}{\left[0, 2\right]} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 8$$

$$x_B = \begin{bmatrix} s_2 \\ x_2 \end{bmatrix} = B^{-1} \overline{b} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Simplex method with bounded variable

In many linear programming problems, we encounter cases where the decision variables have a specific upper or lower limit. In this case, the form of the problem will be as follows:

$$Max \ Z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, ..., m$$

$$x_j \leq u_j, x_j \geq l_j \quad j = 1, ..., n$$

u_j and l_j are fixed numbers that show the maximum increase (upper limit) and the maximum decrease (lower limit) of variable x_j, respectively.

A simple and inefficient way to solve such problems is to deal similarly with bounded variables and other functional constraints of the problem. This way of dealing with problem solving leads to increasing calculations and reducing the efficiency of solving. On the other hand the time required to perform simplex calculations is affected by various factors, the number of constraints being the most important. Experience has shown that the calculation time increases almost by the third power of the number of

constraints. For example, if the number of constraints is doubled, the computation time increases $2^3=8$ times.

The method of solving these problems is divided into two categories of **variables with lower bound** and **variables with upper bound**, which will be examined in the following.

Variables with lower bound

In this section, you will find a situation where the problem only has variables with lower bound. The general form of the problem is as follows:

$$Max\ Z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,...,m$$

$$x_j \ge l_j \quad j = 1,...,n$$

The constraint of the bounded variable can be equalized by subtracting the new variable x_j' from it:

$$x_j - x_j' = l_j \rightarrow x_j = l_j + x_j'$$

By replacing $l_j + x_j'$ with x_j, the lower bound constraint for x_j is removed from the list of functional constraints. The steps for solving problems with lower bound variables are as follows:

Step 1. For variables with lower bounds, change the variable x_j to $l_j + x_j'$ throughout the problem.

Step 2. Solve the new problem resulting from this change of variable without considering the lower bound constraint using the standard simplex method.

Step 3. Convert all x_j' variables obtained in the final table to x_j using the variable change relation.

Example: Consider the following problem.

$$Max \ Z = -x_1 + 4x_2$$
$$-3x_1 + x_2 \le 6$$
$$x_1 + 2x_2 \le 4$$
$$x_2 \ge 1$$
$$x_1, x_2 \ge 0$$

Solve the above model using the lower bound method.

Solution:

x_2 is a variable with a lower limit, that is, the value of x_2 is not less than 1. To remove this constraint, just change the following variable.

$$x_2 = 1 + x_2'$$

As a result, the problem will be as follows:

$$Max \ Z = -x_1 + 4x_2' + 4$$
$$-3x_1 + x_2' \le 5$$
$$x_1 + 2x_2' \le 2$$
$$x_1, x_2' \ge 0$$

Using the simplex method, the optimal solution of the above model is as follows.

$$x_1^* = 0, x_2'^* = 1, x_2^* = x_2'^* + 1 = 2, Z^* = 8$$

Variables with upper bound

In many linear programming problems, the variables of the problem have upper bounds that represent the maximum values of that variable. The upper bound method is a method to solve such problems. The general form of this model is as follows:

$$Max \ Z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i = 1,...,m$$

$$x_j \le u_j \quad j = 1,...,n$$

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The variable with upper bound method is a method that takes the upper bound constraints from functional constraints and treats them separately like non-negative constraints.

The main problem with variables with an upper bound is that changing the variable is not as true as before; because there is no guarantee that changing the variable like below, the value of x_j will remain non-negative.

$$x_j = u_j - x_j' \rightarrow x_j + x_j' = u_j$$

To clarify the above forms, we give an example. Suppose the following constraint is assumed:

$$x_j \leq 3 \rightarrow x_j' = 3 - x_j, x_j' \geq 0$$

For x_j', the value of 5 can be considered because it is a non-negative value, in which case, x_j becomes -2, which is against the assumption of non-negative x_j.

In the simplex method, it is assumed that the values on the right hand side are non-negative, which causes the value of the basic variables to be non-negative or feasible. In the upper bound method, instead of adding the upper bound constraint, its effect can be replaced by changing the feasibility of the basic solution which is no safisfied:

(1) right hand side value is negative or

(2) right hand side value exceed its upper limit.

In the following, the algorithm of the upper bound method is described. More details will be given in solving a problem.

Step 1. Fill the simplex table as normal, but do not enter the upper bound constraints.

Step 2. Select the non-basic variable input to the base, x_k (the variable with the most negative coefficient in the zero row(

Step 3. To select the output variable, calculate the following ratios (a_{ik} are the coefficients of the variable x_k in functional limits).

$$\theta_i = \frac{b_i}{a_{ik}} \quad (i = 1,...,m), a_{ik} > 0$$

$$\alpha_i = \frac{u_i - b_i}{-a_{ik}} \quad (i = 1,...,m), a_{ik} < 0$$

u_k is the upper limit of the input variable. Then calculate the minimum value of $(\theta_i, \alpha_i, u_k)$. Three cases occur:

The first case: the minimum ratio is related to θ_r. Then x_k enters the base and x_r leaves the base. In this case, when x_k enters a basic variable x_r, its value becomes zero. This is the same situation that you encountered before in the standard simplex method.

The second case: the minimum ratio is related to u_k. Then x_k remains non-basic and reaches its upper bound. In this case, the input variable becomes the output variable. The variable (x_k) should be changed to the variable $y_k = u_k - x_k$, then the simplex method is used. With this, the new variable y_k will remain non-primary and with zero value. Because it was possible to enter the base, but it reached its upper bound.

The third case: the minimum ratio is related to α_r. In this case, the basic variable against the negative number of the pivot column (x_r) reaches its upper limit. First a variable replacement is given for x_r as $y_r = u_r - x_r$ and the operation continues according to the simplex method.

Note: The difference between the second and third case is that in the second case, the change of variable is done for the <u>non-basic variable</u> that has the conditions to enter the base, but in the third case, the change of variable is done for the current <u>basic variable</u>.

Step 4: Stop when all numbers in the zero line become non-negative. You have reached the optimal solution.

Example: Consider the following problem. Obtain the optimal solution using the simplex with upper bound method.

$$Max\ Z = x_1 + 2x_2 + x_3$$
$$x_1 + x_2 + x_3 \le 6$$
$$-x_1 + x_2 + x_3 \le 1$$
$$x_1 \qquad \le 10$$
$$x_2 \qquad \le 3$$
$$x_3 \le 4$$
$$x_1, x_2, x_3 \ge 0$$

Solution:

Step 1. By adding slack variables, like the simplex method, the problem enters the following table.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-1	-2	-1	0	0	0
s_1	1	0	1	1	1	1	0	6
s_2	2	0	-1	1	1	0	1	1

Step 2. The variable x2 is chosen as the input variable.

Step 3. To determine the output variable, the following value is calculated.

$$\theta_1 = \frac{6}{1} = 6$$
$$\theta_2 = \frac{1}{1} = 1$$

The value of θ_1 means that the variable x_2 can increase up to 6 units so that the variable s_1 does not become negative. The value of θ_2 means that the variable x_2 can increase by 1 unit so that the variable s_2 does not become negative. This is the concept of ratio test in simplex method.

Because the numbers in the column corresponding to x_2 are positive, there is no need to calculate a. The upper limit of variable x_2 is equal to 3, so we have:

$$Min\{\theta_1,\theta_2,u_2\}=Min\{6,1,3\}=1=\theta_2$$

Since the minimum is related to (the first case of step 3), x_2 is entered and s_2 is removed from the base, and the operation is performed exactly the same as the simplex method.

Basic variable	Row	Z	x_1	x_2	x_3	s_1	s_2	RHS
Z	0	1	-3	0	1	0	2	2
s_1	1	0	2	0	0	1	-1	5
y_2	2	0	-1	-1	1	0	1	-2

Step 3. Since there is a negative number (-3) in the zero row, the problem is non-optimal and x_1 is the input variable. To determine the output variable, the following ratios are calculated.

$$\theta_1=\frac{5}{2}=2.5 \quad for \quad a_{11}=2>0$$
$$\alpha_2=\frac{3-1}{-(-1)}=2 \quad for \quad a_{21}=-1<0$$

So,

$$Min\{\theta_1,\alpha_2,u_1\}=Min\{2.5,2,10\}=2=\alpha_2$$

Note: For simplicity of representation, θ_1 will be the ratio value for row 1, α_2 will be the ratio value for row 2, and u_1 will be the upper bound for the non-basic variable of column 1.

This case represents the second case of step 3. For further explanation, we have the equation of row 2 as follows.

$$-x_1+x_2+x_3+s_2=1$$

The variables x_3 and s_2 are non-basic and their value is zero. Therefore, the above equation is as follows.

$$-x_1 + x_2 = 1 \xrightarrow{x_2 \leq 3} x_1 = x_2 - 1 \leq 3 - 1 = 2 \rightarrow x_1 \leq 2$$

The value of x_1 can increase by two units (a_2) so that the variable x_2 reaches the upper bound which is equal to 3. In this case, x_1 enters the base and x_2 leaves the base and is at its upper bound. In this case, the variable x_2 is changed to $x_2 = 3 - y_2$ so that the variable x_2 is in the upper bound and the variable y_2 has a value of zero and is placed in its lower bound and will be non-basic.

$$Z - 3x_1 + 0 \times x_2 + x_3 + 0 \times s_1 + 2s_2 = 2$$
$$Z - 3x_1 + 0 \times (3 - y_2) + x_3 + 0 \times s_1 + 2s_2 = 2$$
$$Z - 3x_1 \qquad\qquad + x_3 + 0 \times s_1 + 2s_2 = 2$$

Because the coefficient of x_2 in the simplex table is zero, the change of variable has no effect on the row 0 coefficients. There is no change for row 1 in the same way. In the equation of the row 2, there is variable x_2, so we have

$$-x_1 + x_2 + x_3 + s_2 = 1$$
$$-x_1 + (3 - y_2) + x_3 + s_2 = 1$$
$$-x_1 - y_2 + x_3 + s_2 = -2$$

By placing the above equation in the row 2 in the simplex table, we will have

Basic variable	Row	Z	x_1	y_2	x_3	s_1	s_2	RHS
Z	0	1	-3	0	1	0	2	2
s_1	1	0	2	0	0	1	-1	5
y_2	2	0	-1	-1	1	0	1	-2

The variable y_2 in the above table does not have the basic variable condition. Also, the right side of the second line is negative. By multiplying row 2 of this table by -1, the following table is obtained.

Basic variable	Row	Z	x_1	y_2	x_3	s_1	s_2	RHS
Z	0	1	-3	0	1	0	2	2
s_1	1	0	2	0	0	1	-1	5
y_2	2	0	1	1	-1	0	-1	2

Step 3. Since the x_1 coefficient is negative in the zero row, we recalculate the following ratios to determine the output variable. Because 2 and 1 (the numbers under the x_1 variable column) are positive, there is no need to calculate a.

$$\theta_1 = \frac{5}{2} = 2.5$$

$$\theta_2 = \frac{2}{1} = 2$$

$$Min\{\theta_1, \theta_2, u_1\} = Min\{2.5, 2, 10\} = 2 = \theta_2$$

Since the minimum ratio of the result is θ_2, the operation continues as in the simplex method.

Basic variable	Row	Z	x_1	y_2	x_3	s_1	s_2	RHS
Z	0	1	0	3	-2	0	-1	8
s_1	1	0	0	-2	2	1	1	1
x_1	2	0	1	1	-1	0	-1	2

x_3 is the input variable and the following ratios are calculated. Considering that one of the numbers in the x_3 column is positive and the other is negative, the values of a and θ are calculated.

$$\theta_1 = \frac{1}{2} \quad 2 > 0$$

$$\alpha_2 = \frac{10-2}{1} = 8 \quad -1 < 0$$

$$Min\{\theta_1, \alpha_2, u_3\} = Min\left\{\frac{1}{2}, 8, 4\right\} = \theta_1$$

Again, it works like the case 1 of step 3 and the simplex table is as follows.

Basic variable	Row	Z	x_1	y_2	x_3	s_1	s_2	RHS
Z	0	1	0	1	0	1	0	9
x_3	1	0	0	-1	1	0.5	0.5	0.5
x_1	2	0	1	0	0	0.5	-0.5	2.5

The above table has the optimality condition, so the optimal solution is as follows.

$$x_1^* = \frac{5}{2}, y_2^* = 0 \rightarrow x_2^* = 3, x_3^* = \frac{1}{2}, Z^* = 9$$

In the previous problem, the case 2 of step 3 did not occur. In the following example, we examine this case.

Example:

$$Max\ Z = 3x_1 + 5x_2$$
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

Solution:

Considering that the second constraint can be considered as an upper bound for the x_2 variable with a value of 6, therefore, the model can be transferred to the simplex table only by adding a slack variable.

Basic variable	Row	Z	x_1	x_2	s_1	RHS
Z	0	1	-3	-5	0	0
s_1	1	0	3	2	1	18

x_2 is the input variable and $\theta_1 = 9$ and also:

$$Min\{\theta_1, u_2\} = \{9, 6\} = 6 = u_2$$

With the entry of x_2, this non-basic variable reaches its upper bound and the variable must be changed for it: $x_2 = 6 - y_2$. The simplex table is changed based on variable y_2 as follows.

Basic variable	Row	Z	x_1	y_2	s_1	RHS
Z	0	1	-3	5	0	30
s_1	1	0	3	-2	1	6

x_1 will be the input variable and we have:

$$\theta_1 = \frac{6}{3} = 2$$
$$Min\{\theta_1, u_1\} = \{2, 4\} = 2 = \theta_1$$

And the optimal table is as follows.

Basic variable	Row	Z	x_1	y_2	s_1	RHS
Z	0	1	0	3	1	36
x_1	1	0	1	-0.66	0.33	2

The optimal solution will be as follows:

$$x_1^* = 2, y_2^* = 0 \rightarrow x_2^* = 6, Z^* = 36$$

Example: A Company has discontinued the production of a certain unprofitable product line. This act created considerable excess production capacity. The management is considering devoting this excess capacity to one or more of three products; products 1, 2, and 3. The available capacity on the machines is summarized in the following table:

Machine Type	Available time (Machine hours per week)
Milling machine	500
Lather	350
Grinder	150

The number of machine hours required for each unit of the products is:

Machine type	Product 1	Product 2	Product 3
Milling machine	9	3	5
Lathe	5	4	0
Grinder	3	0	2

The sales department indicates that the sales potential for products 1 and 2 exceeds the maximum production rate and that the sales potential for product 3 is 20 units per week. The unit profit would be $50, $20, and $25, respectively, on products 1, 2, and 3. The objective is to determine how much of each product Omega should produce to maximize profit.

a) Formulate a linear programming model for this problem.

b) Solve the model using simplex method.

Solution:

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a) Let x_i be the number of units of product i produced for $i = 1, 2, 3$. The linear programming model of this problem is as follows.

$Max\ Z = 50x_1 + 20x_2 + 25x_3$

$s\,t.$

    (1) $9x_1 + 3x_2 + 5x_3 \leq 500$

    (2) $5x_1 + 4x_2 \quad\quad \leq 350$

    (3) $3x_1 \quad\quad + 2x_3 \leq 150$

    (4) $\quad\quad\quad\quad x_3 \leq 20$

       $x_i \geq 0 \quad i = 1, ..., 3.$

Constraint 1 is the same as milling machine capacity, constraint 2 is the same as lathe capacity, constraint 3 is the same as grinder constraint, and constraint 4 is demand constriant.

b) Solve the model using the simplex method.

$Max\ Z = 50x_1 + 20x_2 + 25x_3$

$s\,t.$

    (1) $9x_1 + 3x_2 + 5x_3 + x_4 = 500$

    (2) $5x_1 + 4x_2 \quad\quad + x_5 = 300$

    (3) $3x_1 \quad\quad + 2x_3 \quad\quad + x_6 = 150$

    (4) $\quad\quad\quad\quad x_3 \quad\quad\quad + x_7 = 20$

       $x_i \geq 0 \quad i = 1, ..., 7.$

Basic variable	Row	Z	X₁	X₂	X₃	X₄	X₅	X₆	X₇	RHS	
Z	0	1	-50	-20	-25	0	0	0	0	0	Non optimal
X₄	1	0	9	3	0	1	0	0	0	500	500/9
X₅	2	0	5	4	0	0	1	0	0	350	350/5
X₆	3	0	3	0	2	0	0	1	0	150	Min 150/3
X₇	4	0	0	0	1	0	0	0	1	20	
Z	0	1	0	-20	-8.33	0	0	16.66	0	2500	Non optimal
X₄	1	0	0	3	-1	1	0	-3.33	0	50	Min 50/3
X₅	2	0	0	4	-3.33	0	1	-1.667	0	100	100/4
X₁	3	0	1	0	0.667	0	0	0.33	0	50	
X₇	4	0	0	0	1	0	0	0	1	20	
Z	0	1	0	0	1.66	6.66	0	-3.3	0	2833.3	Non optimal
X₂	1	0	0	1	-0.33	0.33	0	-1	0	16.6	
X₅	2	0	0	0	-2	-1.33	1	2.33	0	33.3	Min 33.3/2.33
X₁	3	0	1	0	0.66	0	0	0.33	0	50	50/0.33
X₇	4	0	0	0	1	0	0	0	1	20	
Z	0	1	0	0	-1.19	4.76	1.428	0	0	2880.9	Non optimal
X₂	1	0	0	1	-1.19	-0.238	0.428	0	0	30.9	
X₆	2	0	0	0	-0.85	-0.57	0.428	1	0	14.28	
X₁	3	0	1	0	0.95	0.19	-0.142	0	0	45.23	45.23/0.95
X₇	4	0	0	0	1	0	0	0	1	20	Min 20/1
Z	0	1	0	0	0	4.76	1.42	0	1.19	2904	**Optimal**
X₂	1	0	0	1	0	-0.238	0.428	0	1.195	54.76	
X₆	2	0	0	0	0	-0.57	0.42	1	0.85	31.42	
X₁	3	0	1	0	0	0.19	-0.142	0	-0.95	26.19	
X₃	4	0	0	0	1	0	0	0	1	20	

جواب بهینه به صورت $(x_1, x_2, x_3) = (26.19, 54.76, 20)$ و $Z^* = 2904.76$ می‌شود. میزان کمبود و هزینه‌های سایه به صورت زیر می‌شود.

Optimal solution is $(x_1, x_2, x_3) = (26.19, 54.76, 20)$ and $Z^* = 2904.76$. The shadow prices list as follows:

Constraint	The amount of slack or surplus	Shadow price
1	0	4.76
2	0	1.428
3	31.42	0
4	0	1.19

Example: consider the following problem:

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$s\,t.$$

(1) $\quad 3x_1 + 2x_2 + 3x_3 \leq 30$

(2) $\quad 2x_1 + 2x_2 + 3x_3 \leq 40$

$\qquad x_i \geq 0 \quad i = 1,2,3.$

Using the simplex method to solve the problem.

Solution:

After the slack variables needed for the inequality constraints are introduced, the original model becomes:

$$Max\ Z = 4x_1 + 3x_2 + 6x_3$$
$$s\,t.$$

(1) $\quad 3x_1 + 2x_2 + 3x_3 + x_4 \quad\ = 30$

(2) $\quad 2x_1 + 2x_2 + 3x_3 \quad\ + x_5 = 40$

$\qquad x_i \geq 0 \quad i = 1,...,5.$

Now we can apply the simplex method as follows:

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	RHS	
Z	0	1	-4	-3	-6	0	0	0	Non-optimal
X_4	1	0	3	2	3	1	0	30	Min 30/3
X_5	2	0	2	2	3	0	1	40	40/3
Z	0	1	2	1	0	2	0	60	Optimal
X_3	1	0	1	0.66	1	0.333	0	10	
X_5	2	0	-1	0	0	-1	1	10	

The optimal solution is $X = (0,0,10,0,10)$ and $Z^* = 60$.

Example: consider the following problem:

$Max\ Z = 3x_1 + x_2 + 5x_3 + 4x_4$

$s.t.$

     (1) $3x_1 - 3x_2 + 2x_3 + 8x_4 \leq 50$

     (2) $5x_1 + 6x_2 - 4x_3 - 4x_4 \leq 40$

     (3) $4x_1 - 2x_2 + x_3 + 3x_4 \leq 20$

         $x_i \geq 0 \quad i = 1,2,3,4.$

Work through the simplex method step by step to demonstrate that Z is unbounded.

Solution:

$Max\ Z = 3x_1 + x_2 + 5x_3 + 4x_4$

$s.t.$

     (1) $3x_1 - 3x_2 + 2x_3 + 8x_4 + x_5 = 50$

     (2) $5x_1 + 6x_2 - 4x_3 - 4x_4 + x_6 = 40$

     (3) $4x_1 - 2x_2 + x_3 + 3x_4 + x_7 = 20$

         $x_i \geq 0 \quad i = 1,...,7.$

Now we can apply the simplex method as follows:

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	X_6	X_7	RHS	
Z	0	1	-3	-1	-5	-4	0	0	0	0	Non-optimal
X_5	1	0	3	-3	2	8	1	0	0	50	25=50/2
X_6	2	0	5	6	-4	-4	0	1	0	40	
X_7	3	0	4	-2	1	3	0	0	1	20	Min 20=20/1
Z	0	1	17	-11	0	11	0	0	5	100	Non-optimal
X_5	1	0	-5	1	0	2	1	0	-2	10	Min 10/1
X_6	2	0	21	-2	0	8	0	1	4	120	
X_3	3	0	4	-2	1	3	0	0	1	20	
Z	0	1	-38	0	0	33	11	0	-17	210	Non-optimal
X_2	1	0	-5	1	0	2	1	0	-2	10	
X_6	2	0	11	0	0	12	2	1	0	140	Min 140/11
X_3	3	0	-6	0	1	7	2	0	-3	40	
Z	0	1	0	0	0	74.45	17.9	3.45	-17	693	Unbounded
X_2	1	0	0	1	0	7.45	1.9	0.45	-2	73.63	
X_1	2	0	1	0	0	1.09	0.18	0.09	0	12.72	
X_3	3	0	0	0	1	13.54	3.09	0.545	-3	116.36	

در جدول فوق، ستون زیر متغیر x_7 ، نامثبت هستند و لذا شروط نامحدود بودن برقرار می‌شــود و بــا افزایش دو متغیر x_2 و x_3 مقدار تابع هدف به سمت بی نهایت می رود.

In the above table, the columns below the variable x_7 are non-positive demonstrating that Z is unbounded. In fact, by increasing the two variables x_2 and x_3, the value of the objective function goes to infinity.

Example: Consider the following problem:

$Min\ Z = 0.4x_1 + 0.5x_2$

$s.t.$
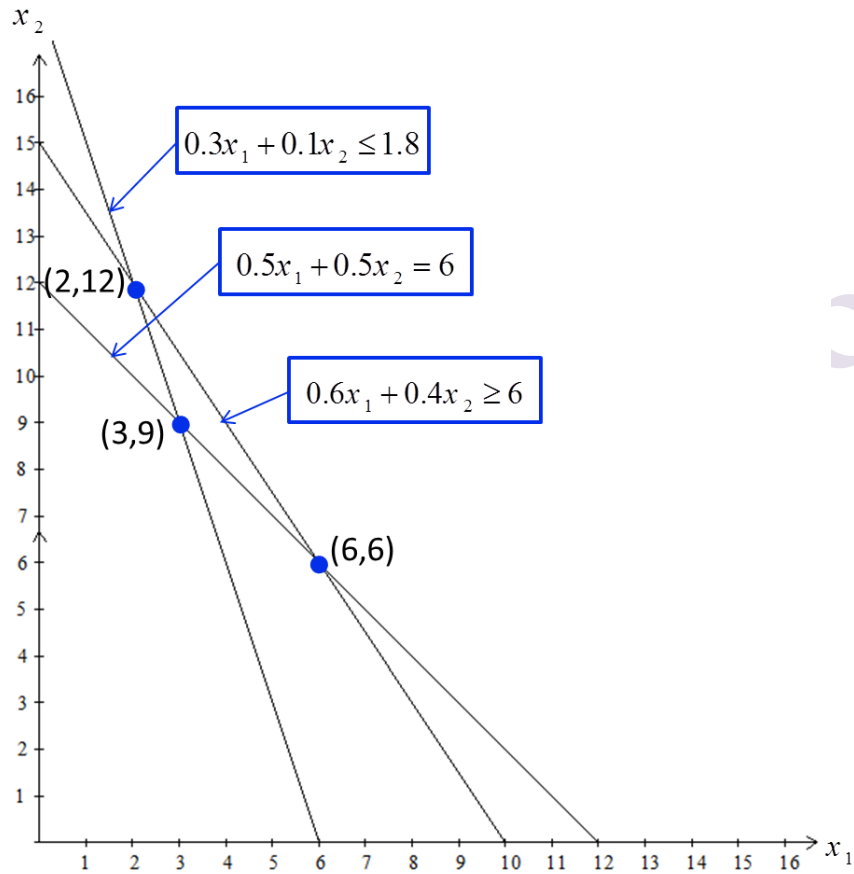
(1) $0.3x_1 + 0.1x_2 \le 1.8$

(2) $0.5x_1 + 0.5x_2 = 6$

(3) $0.6x_1 + 0.4x_2 \ge 6$

     $x_i \ge 0 \quad i = 1, 2.$

a) Solve this problem graphically.

b) Using the Big M method, work through the simplex method step by step to solve the problem.

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Solution:

a) The model has no feasible region.

b) convert min to max, and use artificial/surplus/slack variables and the Big M method to construct the complete first simplex tableau for the simplex method,

$$Max \ -Z + 0.4x_1 + 0.5x_2 + Mx_4 \ + Mx_6$$
$$s\,t.$$

(1) $0.3x_1 + 0.1x_2 + x_3 \qquad\qquad\qquad = 1.8$

(2) $0.5x_1 + 0.5x_2 \qquad + x_4 \qquad\qquad = 6$

(3) $0.6x_1 + 0.4x_2 \qquad\qquad - x_5 + x_6 = 6$

$\qquad x_i \geq 0 \quad i = 1, 6.$

Basic variable	Row	Z	X_1	X_2	X_3	X_4	X_5	X_6	RHS	
Z	0	-1	-1.1M+0.4	-0.9M+0.5	0	0	M	0	-12M	Non-optimal
X_3	1	0	0.3	0.1	1	0	0	0	1.8	Min 1.8/0.3
X_4	2	0	0.5	0.5	0	1	0	0	6	6/0.5
X_6	3	0	0.6	0.4	0	0	-1	1	6	6/0.5
Z	0	1-	0	-0.533M +0.367	3.67M-1.33	0	M	0	-5.4M-2.4	Non-optimal
X_1	1	0	1	0.33	3.33	0	0	0	6	6/0.33
X_4	2	0	0	0.33	-1.667	1	0	0	3	Min 3/0.33
X_6	3	0	0	0.2	-2	0	-1	1	2.4	2.4/0.2
Z	0	-1	0	0	M+0.5	1.6M-1.1	M	0	-0.6M-5.7	Optimal
X_1	1	0	1	0	5	-1	0	0	3	
X_2	2	0	0	1	-5	3	0	0	9	
X_6	3	0	0	0	-1	-0.6	-1	1	0.6	

In the final table, the optimal condition is reached, but $x_6 = 0.6$ is greater than zero, which means that the original model has no solution.