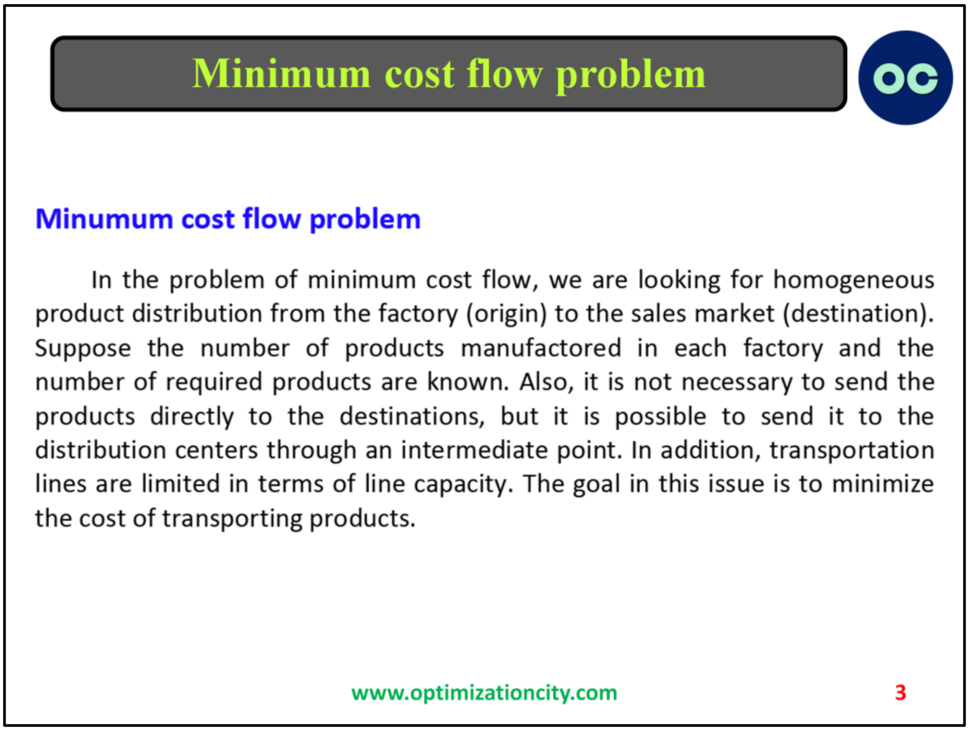
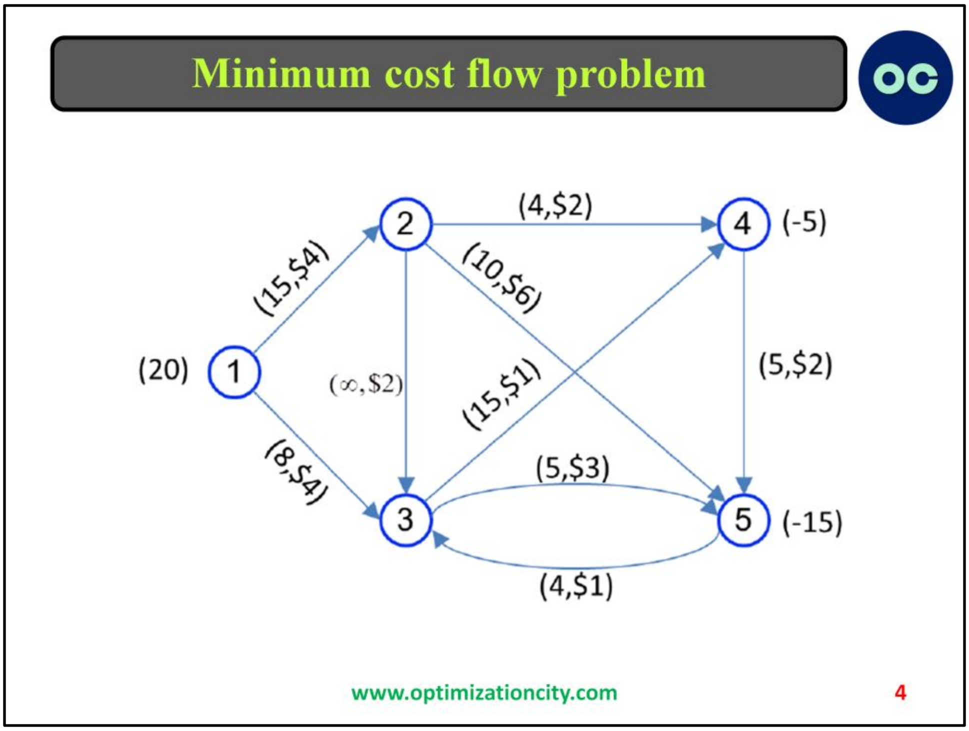
Welcome to the seventh course on our website, where we will explore the concept of Minimum Spanning Trees! In this course, we'll dive into the idea of finding the most efficient way to connect a set of points while minimizing the total cost or distance.

فهرست مطالب

In this course, we will cover important topics related to graph optimization. We start by introducing the minimum cost flow problem, explaining its basics and why it matters in different areas. Next, we explore the concept of minimum spanning trees, discussing their features, practical uses, and how they help solve connectivity issues. Then, we dive into the algorithms used to find minimum spanning trees. Finally we do some examples.



In the minimum cost flow problem, we want to distribute the same type of product from the factory to the sales market efficiently. We know the quantity of products produced at each factory and the required quantity at each destination. Instead of sending products directly to the destinations, we can send them to distribution centers through an intermediate point if needed. However, there are limits on how much can be transported on each transportation line. The objective is to minimize the cost of transporting the products while meeting the demand.



Let's look at a simple example of the minimum cost flow problem using the diagram below. The nodes are shown as numbered circles, and the arcs are represented by arrows. The arcs have a specific direction, meaning materials or goods can be transported from one node to another in that direction. For instance, we can send materials from node 1 to node 2, but we cannot send them from node 2 to node 1. We use the notation i-j to indicate the arc going from node i to node j.

In the diagram above, each arrow represents a path with a specific capacity and cost for transportation. The capacity indicates the maximum amount that can be transported along that path, while the cost per unit represents the expense associated with each unit of transportation. For instance, in the arrow (2-4), the flow can range from 0 to 4 units, and each unit passing through this path costs $2. The symbol ∞ indicates that there is no limit on the capacity of that particular path.

Moreover, the numbers in parentheses next to the circles represent the available supply and the required demand at each node. In this diagram, node 1 is the starting point with a supply of 20 units. Nodes 4 and 5 are the destinations that require 5 and 15 units respectively, indicated by the negative sign.

Minimum cost flow problem

x_{ij}: is the number of units transported from node i to node j using arc i-j.

$$
\begin{array}{lllllllllll}
Min & 4x_{12} + 4x_{13} + 2x_{23} + 2x_{24} + 6x_{25} + x_{34} + 3x_{35} + 2x_{45} + x_{53} \\
s.t. \\
(1) & x_{12} + x_{13} & & & & & & & & & = 20 \\
(2) & -x_{12} & + x_{23} & + x_{24} + x_{25} & & & & & & & = 0 \\
(3) & & -x_{13} - x_{23} & & + x_{34} & + x_{35} & & - x_{53} & & & = 0 \\
(4) & & & -x_{24} & - x_{34} & & + x_{45} & & & & = -5 \\
(5) & & & -x_{25} & & - x_{35} - x_{45} & + x_{53} & & & & = -15
\end{array}
$$

$$x_{12} \le 15; x_{13} \le 8; x_{23} \le \infty; x_{24} \le 4; x_{25} \le 10; x_{34} \le 15; x_{35} \le 5; x_{45} \le \infty; x_{53} \le 4.$$

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In the minimum cost flow problem, our objective is to discover the flow pattern that incurs the lowest cost. To formulate this problem as a linear programming model, let's consider the following notation:
xij: represents the quantity of units transported from node i to node j using the arc i-j.
Now, let's present the linear programming model for the minimum cost flow problem.

Equations 1 to 5 represent the flow balance equations in the network. Let's consider the balance equation at node 1 as an example.
The equation states that the combined flow leaving node 1 (x12 + x13) should be equal to the supply available at node 1, which is 20 units. This ensures that the outgoing flow matches the available supply.

Similarly, the balance equation at node 2 indicates that the incoming flow to node 2 (x12) is equal to the outgoing flow from node 2 (x23 + x24 + x25). This ensures that the flow entering and leaving node 2 is properly balanced.

The minimum cost flow model in the network has a unique structure that helps determine the solution algorithm. The flow variables, denoted as xij in the balance equations, have coefficients of 0, +1, or -1. Additionally, these variables appear precisely in two balance equations: once with a coefficient of +1 representing the node where the flow originates, and once with a coefficient of -1 representing the

node where the flow enters.
Based on the above characteristics, the general form of the Minimum Cost Flow problem can be expressed as follows.

Minimum cost flow problem

$$Min \quad \sum_i \sum_j c_{ij} x_{ij}$$

$s.t.$

$$\sum_j x_{ij} - \sum_k x_{ki} = b_i \quad i = 1, \dots, n$$

$$l_{ij} \le x_{ij} \le u_{ij}$$

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The model shown here represents the general form of the minimum cost flow problem. However, we can simplify this model into more straightforward forms, which are explained below.

Minimum spanning tree problem

Minimum spanning tree

In this problem, it is considered to connect different nodes of a network to each other so that the minimum distance is established. Arcs connecting nodes (points) are undirected. Some examples of the application of this problem are:

1- A company in the construction of its central building wants to use the least amount of electric cable for the supply of electricity to various points in that building.

2- The Ministry of Roads is looking for a way to connect the villages of the same region based on which the length of the connecting roads is minimal.

3- The Water and Sewerage Department intends to use the shortest length of pipe to create a water supply network in a town under construction.

The minimum spanning tree problem involves connecting different nodes in a network in such a way that the shortest distance is established. The connections between nodes are not directional. Here are some examples of how this problem can be applied:
1- A company needs to minimize the amount of electric cable used to provide electricity to various points in their central building during construction.
2- The Ministry of Roads is seeking the most efficient way to connect villages in the same region, minimizing the length of the connecting roads.
3- The Water and Sewerage Department aims to create a water supply network in a town under construction using the shortest possible length of pipe.

Minimum spanning tree algorithm

Minimum spanning tree algorithm

Assuming that there are n number of points (nodes) numbered from 1 to n in the given problem, the steps of the algorithm are as follows:

Step 1: Form an n*n square matrix of distances between points 1 to n.

Step 2: Start from an arbitrary point and find the shortest distance between that point and other points. For this purpose, the row corresponding to the selected point should be taken out of the main matrix and form a 1*n matrix and choose the smallest number in this matrix. In this case, the selected number is the shortest distance of the selected point. Consider the latter point as the second point.

Step 3: In this step, form the 2*n matrix related to the two indicated points and select the smallest number inside the matrix to get the third point as well. Continue like this until all points are selected.

The algorithm for finding the minimum spanning tree can be explained using simple words as follows:
Step 1: Create a square matrix of size n*n, representing the distances between the n points (nodes) numbered from 1 to n.
Step 2: Start from any point and find the shortest distance between that point and the other points. To do this, extract the row corresponding to the selected point from the matrix and create a 1*n matrix. Choose the smallest number in this matrix as the shortest distance. Consider the corresponding point as the second point.
Step 3: Repeat the process in Step 2, but now consider the two points selected so far. Create a 2*n matrix based on these two points and select the smallest number in the matrix as the third point. Continue this process until all points are selected.
Step 4: The order in which the points are obtained represents the optimal sequence for connecting them in the tree. The sum of the distances in this tree will be the minimum possible distance to connect all the points.
Note: If there are two or more equal numbers in the matrix at any stage, there may be multiple solutions to the problem.

Minimum spanning tree algorithm

Step 4: The order of obtaining these points indicates the order of optimal movement in the tree, and the sum of the distances in this tree will be the minimum possible distance to connect the points.

Note: If there are two or more equal numbers in the matrix at a stage, there will be two or more solutions to the problem.

Step 4: The order in which the points are obtained represents the optimal sequence for connecting them in the tree. The sum of the distances in this tree will be the minimum possible distance to connect all the points.
Note: If there are two or more equal numbers in the matrix at any stage, there may be multiple solutions to the problem.

Exercises

Example:

	1	2	3	4	5	6
1	–	1	6	7	9	11
2	1	–	6	4	3	∞
3	5	5	–	5	∞	6
4	7	4	6	–	6	3
5	9	3	∞	6	–	∞
6	11	∞	7	3	∞	–

In a building, there are six points that need electrical wiring. The distances between these points are given in a matrix. The symbol 'd' represents the impossibility of wiring between two points. By applying the minimum spanning tree algorithm, we can find the minimum wiring tree and determine the minimum amount of wires needed for the connections.

Exercises

Solution:

In stage 1 (k=1)

$$k=1 \rightarrow \begin{array}{c|cccccc} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} \\ \underline{1} & - & \boxed{1} & 6 & 7 & 9 & 11 \end{array}$$

In stage 2 (k=2),

$$k=2 \rightarrow \begin{array}{c|cccccc} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} \\ \underline{1} & - & - & 6 & 7 & 9 & 11 \\ \underline{2} & - & - & 6 & 4 & \boxed{3} & \infty \end{array}$$

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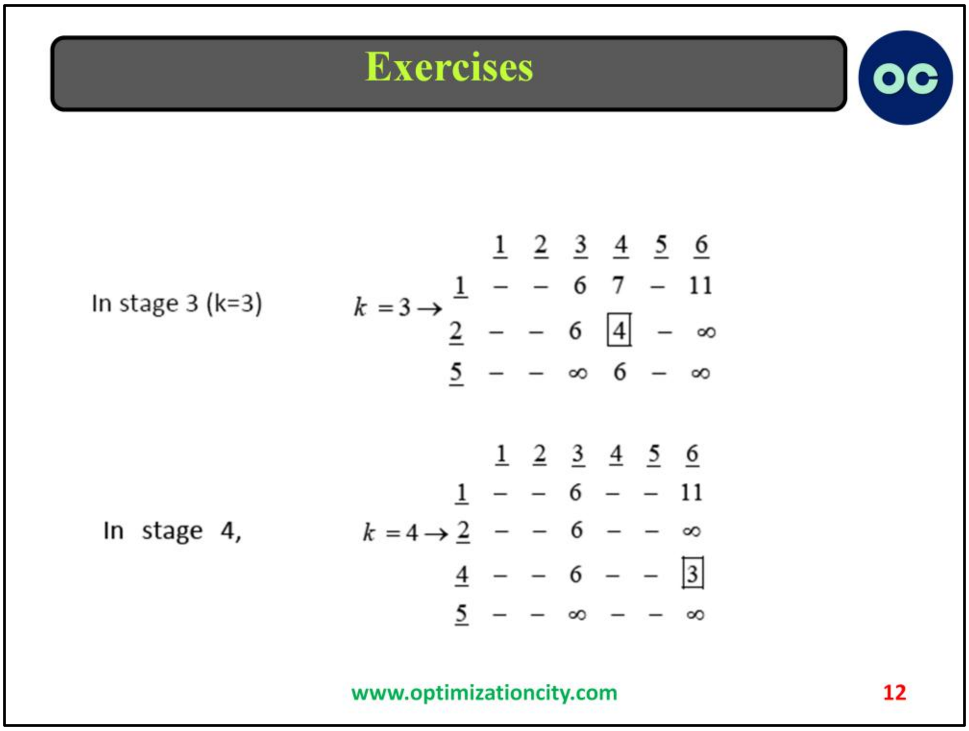
In the first stage (k=1), we need to choose a point arbitrarily. Let's choose point 1 (although any other point could also be chosen). We create a 1*6 matrix for this point as follows:

In this matrix, the smallest number is 1, which is marked with a square around it. This number represents the distance between point 1 and point 2, indicating that the shortest distance to point 2 has been found.

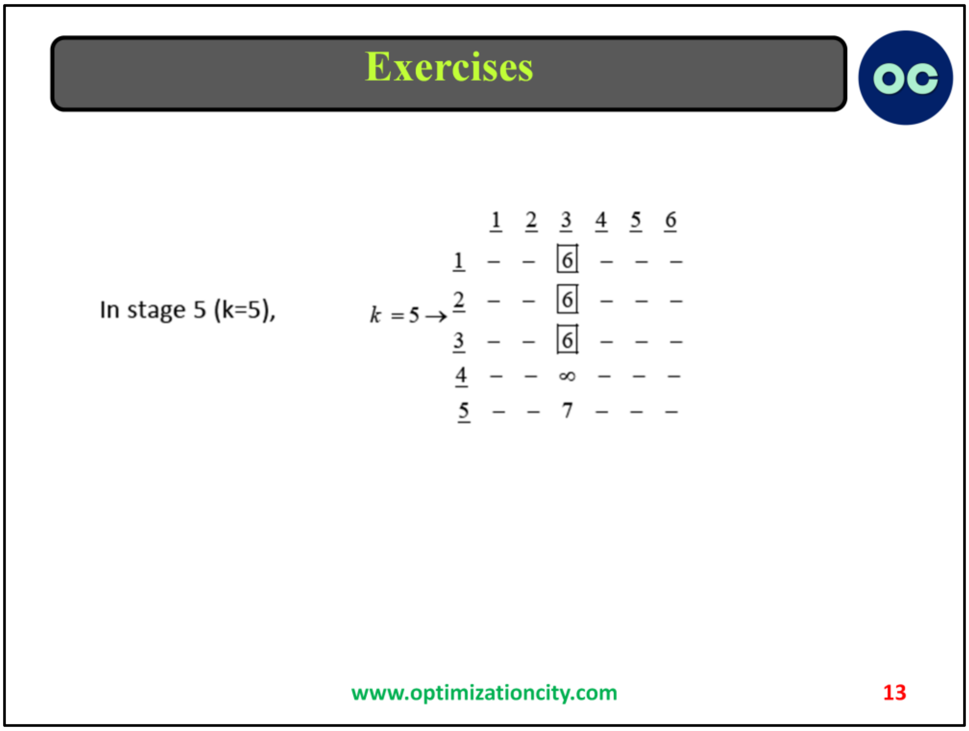
In the second stage (k=2), we add point 2, which had the shortest distance in the previous step, to the 1*6 matrix. This creates a new 2*6 matrix:

Note that since points 1 and 2 have been assigned, we mark columns 1 and 2 with a dash symbol, indicating that there is no need to check these points. In this matrix, the smallest number is 3, which represents the distance between point 2 and point 5. Therefore, we have also found the shortest distance to point 5.



In the third stage (k=3), we expand the matrix by adding row 5. This results in a new 3*6 matrix.

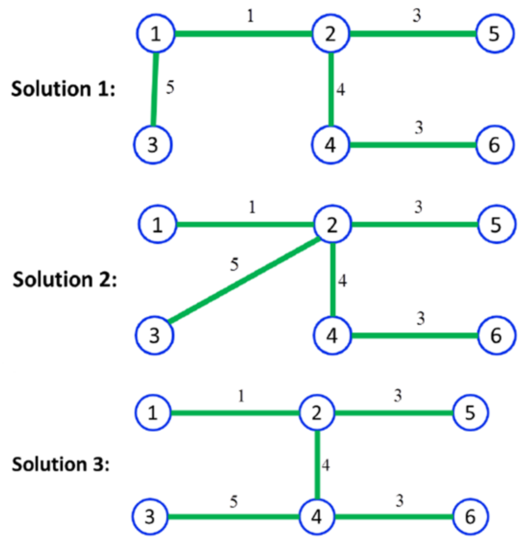
In the fourth stage, we identify the smallest number in this matrix, which is 4. This number represents the distance between point 2 and point 4, resulting in the shortest distance to point 4. We continue performing the same steps for the subsequent stages, generating the following matrices:



The matrix in the fifth stage (k=5) is shown below:

In this stage, we observe that there are three numbers equal to 6, which are the smallest numbers in this matrix. As a result, the problem has three solutions. This means that for wiring electricity to point 3, we can start from either point 1, point 2, or point 4. Since the shortest distance to all points has been obtained, the algorithm concludes. The figure below illustrates the three solutions to the problem, which represent the minimum spanning tree.

Exercises



Exercises

Example:

Obtain the minimum spanning tree with the following distance matrix.

	1	2	3	4	5	6
1	–	5	9	3	7	5
2	5	–	7	3	5	6
3	9	7	–	6	5	8
4	3	3	6	–	8	4
5	7	5	5	8	–	6
6	5	6	8	4	6	–

Example:
Let's find the minimum spanning tree using the given distance matrix.

Exercises

Solution:

We choose node 6 arbitrarily. The arc (6,4) is chosen with the lowest cost.

$k=1 \rightarrow$

	1	2	3	4	5	6
6	5	6	8	$\boxed{4}$	6	–

Nodes 1 and 2 are selected and arcs (4,1) and (4,2) are added to the tree with the lowest cost.

$k=2 \rightarrow$

	1	2	3	4	5	6
4	$\boxed{3}$	$\boxed{3}$	6	–	8	–
6	5	6	8	–	6	–

Solution:
We start by choosing node 6 as an arbitrary starting point. The arc (6,4) is selected with the lowest cost.
Next, we select nodes 1 and 2, and add arcs (4,1) and (4,2) to the tree with the lowest cost.

Exercises

		1	2	3	4	5	6
	1	–	–	9	–	7	–
$k=3,4 \rightarrow$	2	–	–	7	–	$\boxed{5}$	–
	4	–	–	6	–	8	–
	6	–	–	8	–	6	–

		1	2	3	4	5	6
	1	–	–	9	–	–	–
$k=5 \rightarrow$	2	–	–	7	–	–	–
	4	–	–	6	–	–	–
	5	–	–	$\boxed{5}$	–	–	–
	6	–	–	8	–	–	–

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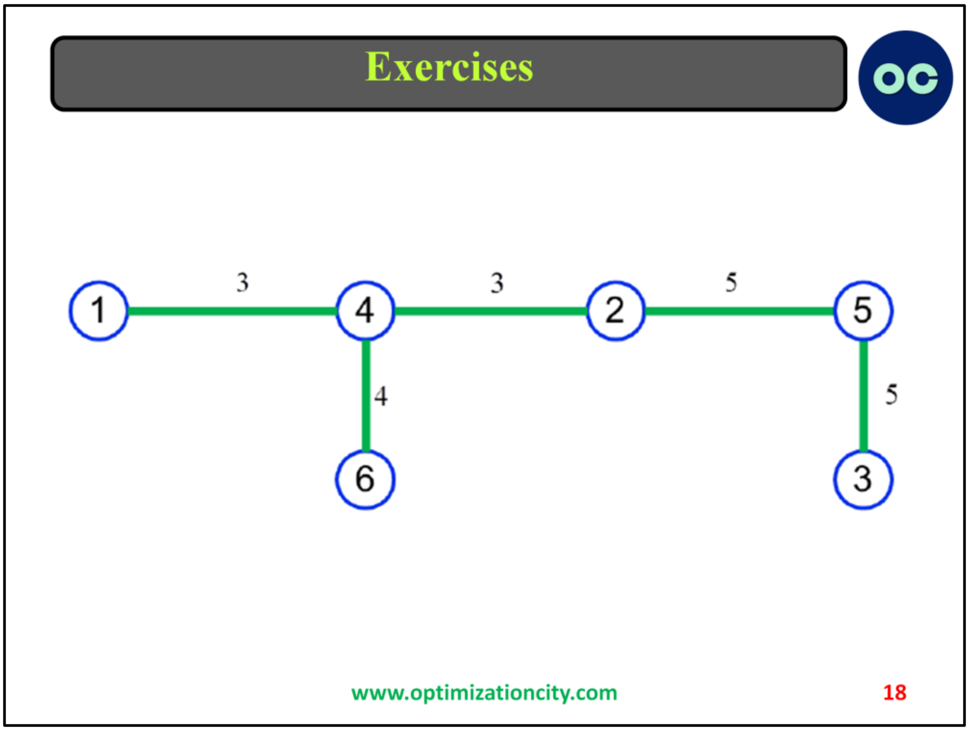
Then, we choose node 5 and add the arc (2,5) to the tree with the lowest cost.

Finally, we select node 3 and add the arc (5,3) to the tree with the lowest cost.



The resulting minimum spanning tree is as follows.

If you want more practice to improve what you've learned, I've got something perfect for you. Check out my website where you'll find plenty of extra exercises to explore. Take a moment to visit my website and try out the exercises. They're not only educational but also enjoyable. I'm confident they'll help you strengthen your understanding and feel more confident. Give it a try and see for yourself.

Thanks

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