OptimizationCity
Encyclopedia of Operations Research

# Course 4: Parametric linear programming

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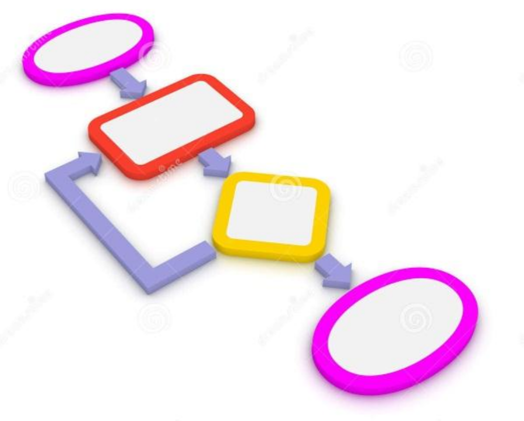
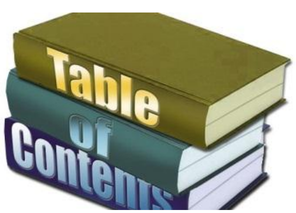


# فهرست مطالب





# Sensitivity analysis

## Sensitivity analysis

Sensitivity analysis is a procedure that is implemented after obtaining the optimal solution. Sensitivity analysis determines the sensitivity of the optimal solution against certain changes in the original model.

$a_{ij}, c_j, b_i$ are certain! It is not true in the real world.

# Sensitivity analysis

Changes that are usually studied in the linear programming model include the following:

1) Changes in the numbers on the right hand side

2) Changes in the coefficients of the objective function

3) Add a new constraint

# Sensitivity analysis

the result of these changes

1) The optimal answer remains unchanged, that is, the basic variables and their values do not change at all.

2) Basic variables should not change, but their values should change.

3) The basic solution should be completely changed.

The first case shows the insensitivity of the optimal solution to model changes and the third case shows more sensitivity.

# Sensitivity analysis of right hand side

Sensitivity analysis of right hand side

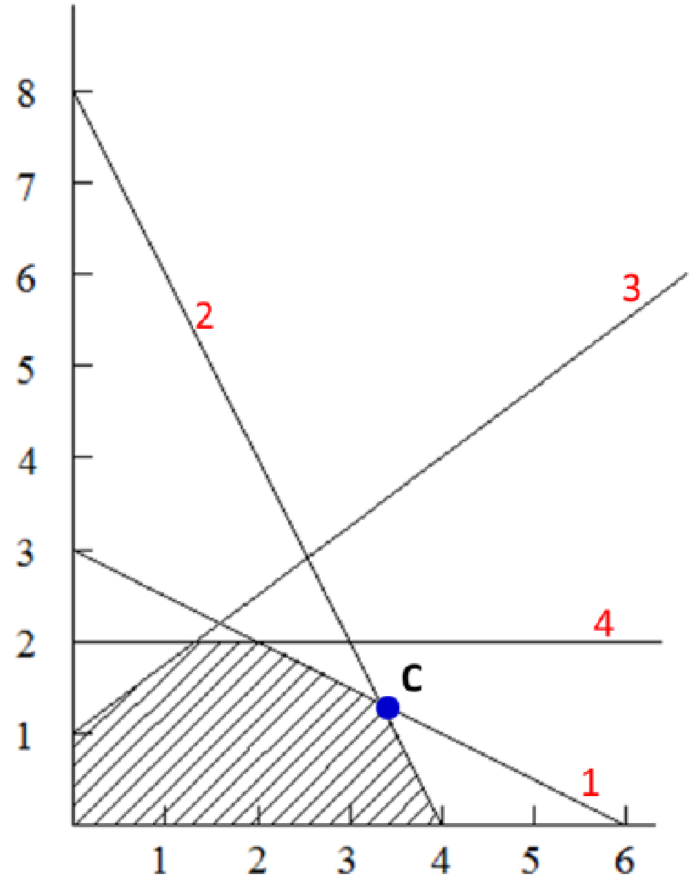
# Sensitivity analysis of right hand side

**Example:**

In the following problem, in order to produce two products whose production values are represented by $x_1$ and $x_2$, four resources a, b, c, and d are used, which are on the right hand side of the constraint 1, 2, 3, and 4, respectively. The amount of available resources is equal to a=6, b=8, c=1, d=2, and the four constriants of the problem are written based on the limitations of these four resources. If the profit per unit of the first and second product is 3 and 2 respectively, the model is as follows.

$$
\begin{aligned}
Max \quad Z &= 3x_1 + 2x_2 \\
x_1 + 2x_2 &\le 6 \\
2x_1 + x_2 &\le 8 \\
-x_1 + x_2 &\le 1 \\
x_2 &\le 2 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

# Sensitivity analysis of right hand side




$$x_1^* = 3\frac{1}{3}, x_2^* = 1\frac{1}{3}, s_1^* = 0, s_2^* = 0, s_3^* = 3, s_4^* = \frac{2}{3}$$

# Sensitivity analysis of right hand side

**How much can the amount of resources (numbers on the right) decrease or increase?**

In particular, we are interested in two types of analysis:

1) In order to improve the optimal value of the objective function, how much of a resource can be increased?

2) How much of a resource can be reduced without causing a change in the current optimal solution?

Since the source level is expressed by the numbers on the right side of the constraints, this type of analysis is called the **sensitivity analysis of the right hand side**.

# Sensitivity analysis of right hand side

$$Max \quad Z = 3x_1 + 2x_2$$

$$x_1 + 2x_2 + s_1 = 6$$

$$2x_1 + x_2 + s_2 = 8$$

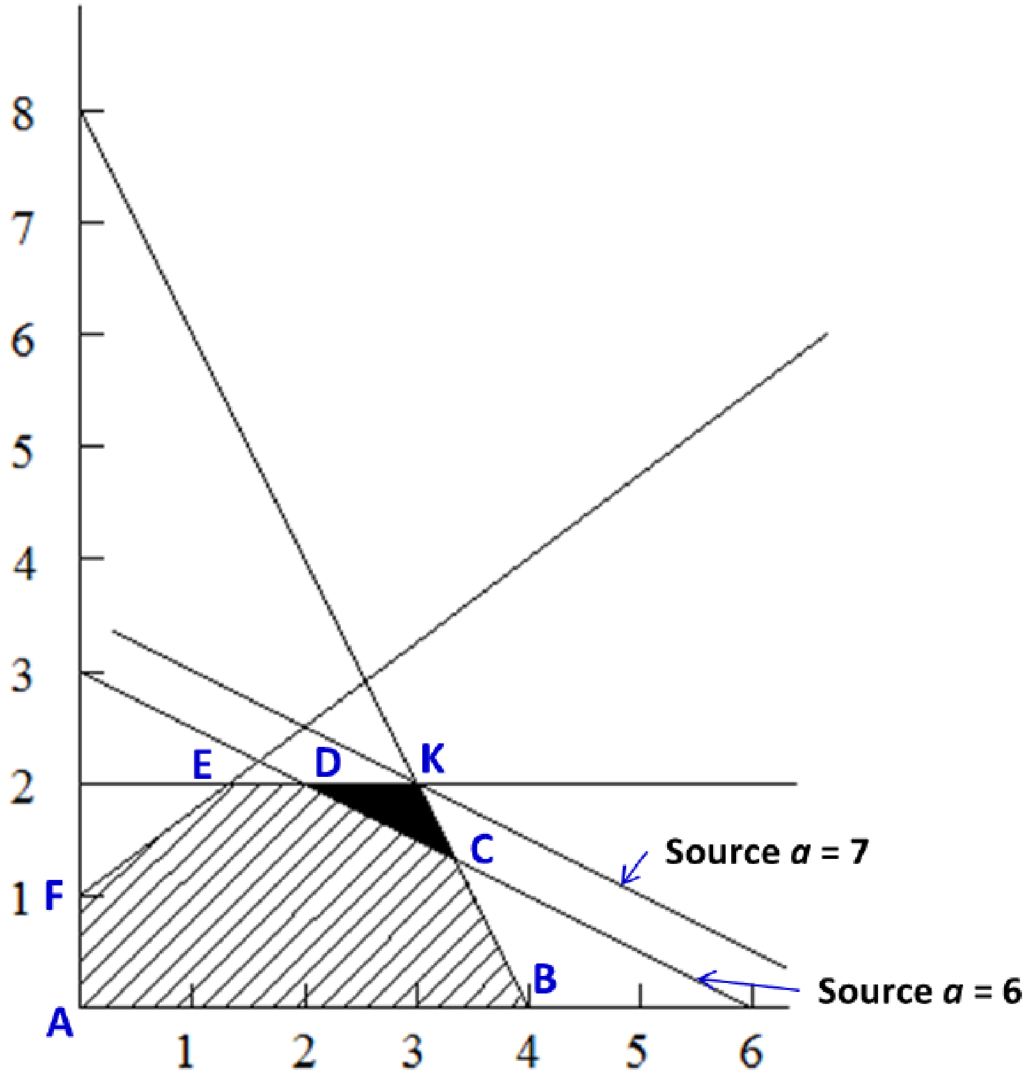
$$-x_1 + x_2 + s_3 = 1$$

$$x_2 + s_4 = 2$$

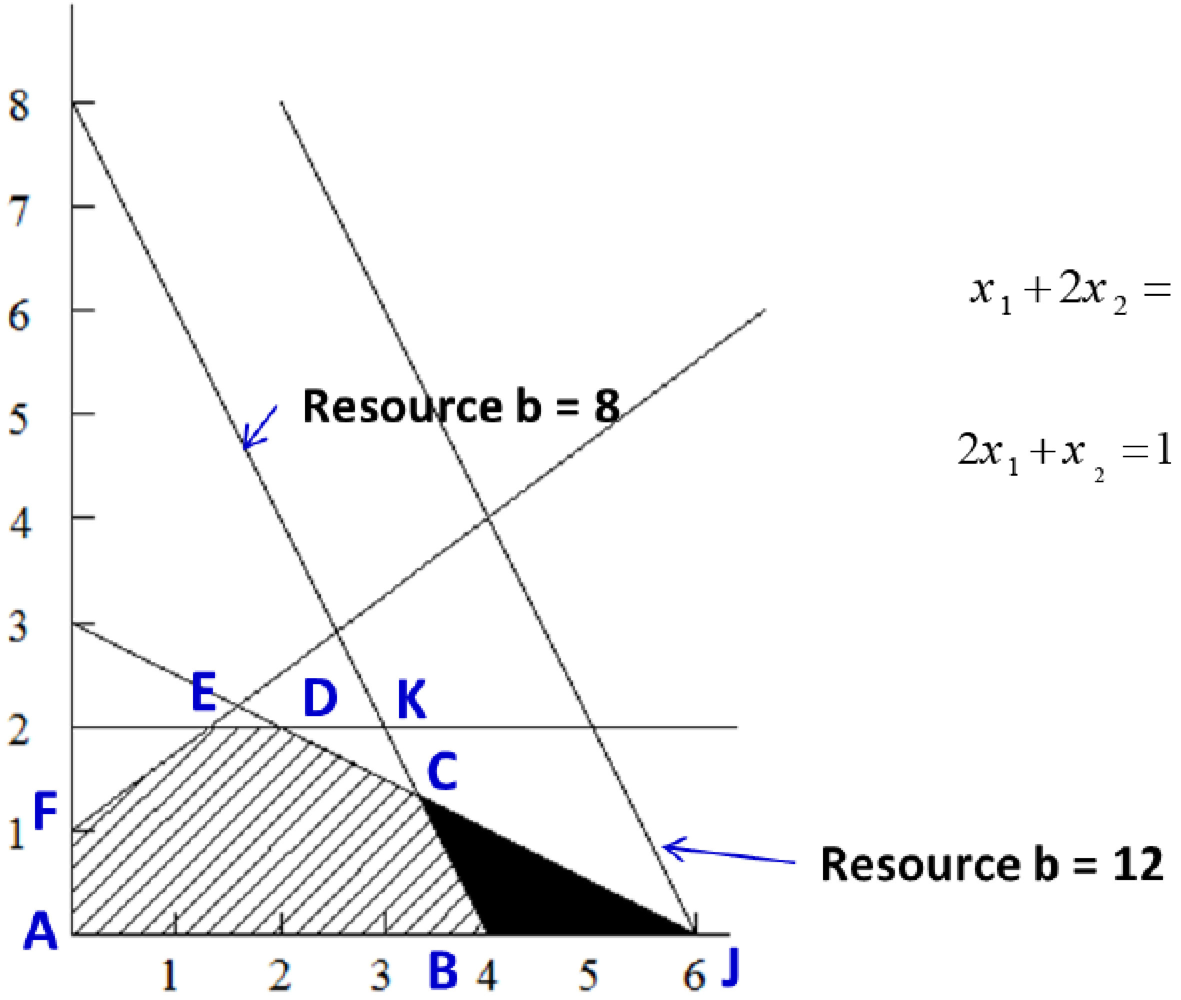
$$x_1, x_2 \geq 0$$

$$x_1^* = 3\frac{1}{3}, x_2^* = 1\frac{1}{3}, s_1^* = 0, s_2^* = 0, s_3^* = 3, s_4^* = \frac{2}{3}$$

# Sensitivity analysis of right hand side



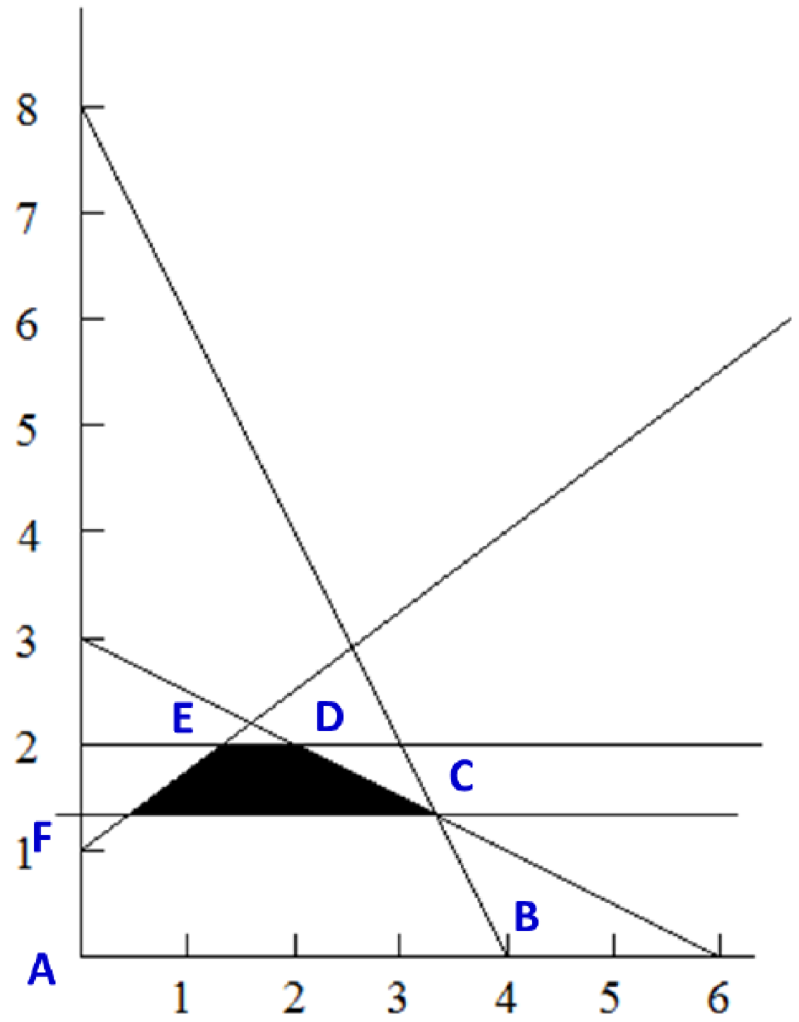
# Sensitivity analysis of right hand side




$$x_1 + 2x_2 = 6, x_2 = 0 \rightarrow x_1 = 6, x_2 = 0$$

$$2x_1 + x_2 = 1 \rightarrow 2(6) + (1)(0) = 12$$

# Sensitivity analysis of right hand side



# Sensitivity analysis of right hand side

Now consider the third constraint. Again, the right hand side of the constraint can be reduced so that the equation corresponding to the third constraint ($-x_1+x_2=1$) passes through point C. Therefore, the right hand side of the third constraint is equal to $-x_1+x_2=\left(-3\frac{1}{3}\right)+\left(1\frac{1}{3}\right)=-2$. This change does not affect the current optimal point, C. The results of the above discussions are summarized in the table below.

# Sensitivity analysis of right hand side

| Resource | Type | Maximum change in Z value | Maximum change in resource value |
|---|---|---|---|
| 1 | Scarce | $13-12\frac{2}{3}=\frac{1}{3}$ | $7-6=1$ |
| 2 | Scarce | $18-12\frac{2}{3}=5\frac{1}{3}$ | $12-8=4$ |
| 3 | Redundant | $12\frac{2}{3}-12\frac{2}{3}=0$ | $-2-1=-3$ |
| 4 | Redundant | $12\frac{2}{3}-12\frac{2}{3}=0$ | $1\frac{1}{3}-2=-\frac{2}{3}$ |

# Sensitivity analysis of right hand side

| Resource | Type | Value of $y_i$ |
| --- | --- | --- |
| 1 | Scarce | $y_1 = \frac{1}{3}$ |
| 2 | Scarce | $y_2 = \frac{4}{3}$ |
| 3 | Redundant | $y_3 = 0$ |
| 4 | Redundant | $y_4 = 0$ |

$$y_i = \frac{Z^* \text{ Maximum change in optimum value}}{b_i \text{ Allowable increase in source i}}$$

# Sensitivity analysis of objective function

## Sensitivity analysis of objective function coefficients

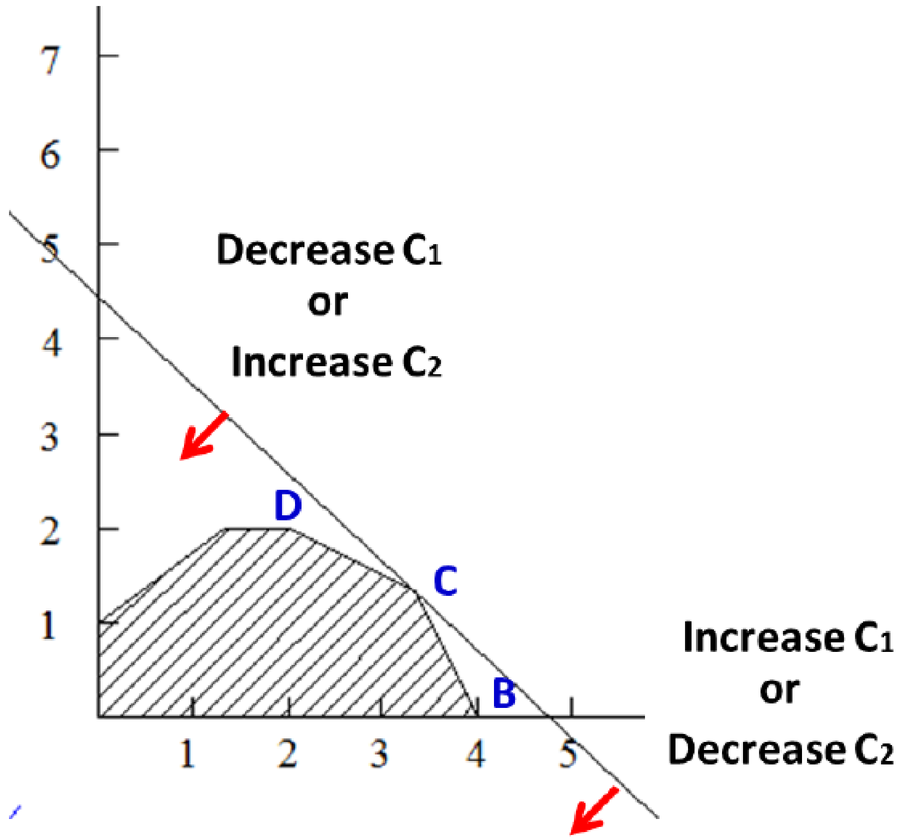
What is the question?

# Sensitivity analysis of objective function

**Example:**

If the coefficients of the objective function in the previous example are denoted by $C_1$ and $C_2$, the objective function becomes:

$$Z = C_1x_1 + C_2x_2$$



# Sensitivity analysis of objective function

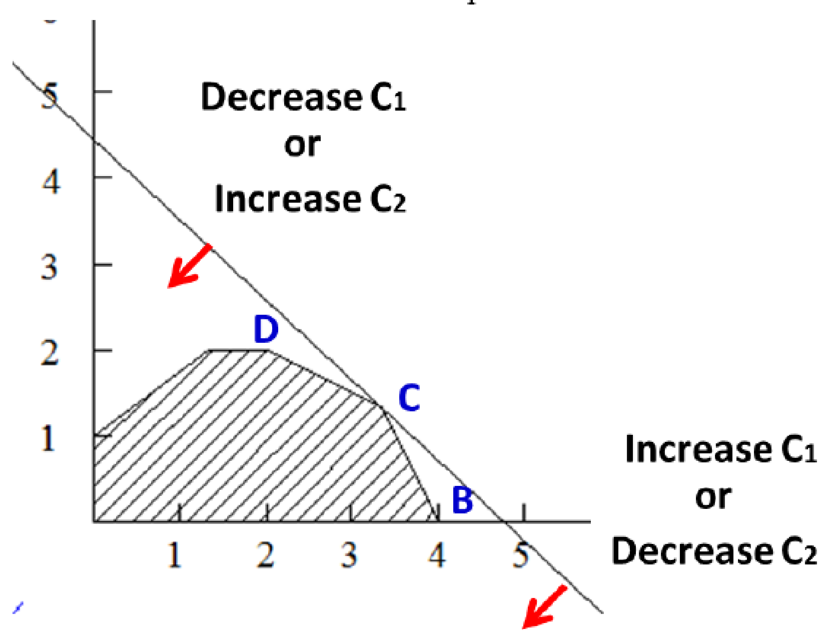
$$-\frac{C_1}{2} = \frac{-1}{2} \rightarrow C_1 = 1$$

$$-\frac{C_1}{2} = \frac{-2}{1} \rightarrow C_1 = 4$$

$$\Rightarrow \quad 1 \le C_1 \le 4$$

The range of changes of $C_1$ to remain optimal at point C is as follows:

$$1 \le C_1 \le 4$$



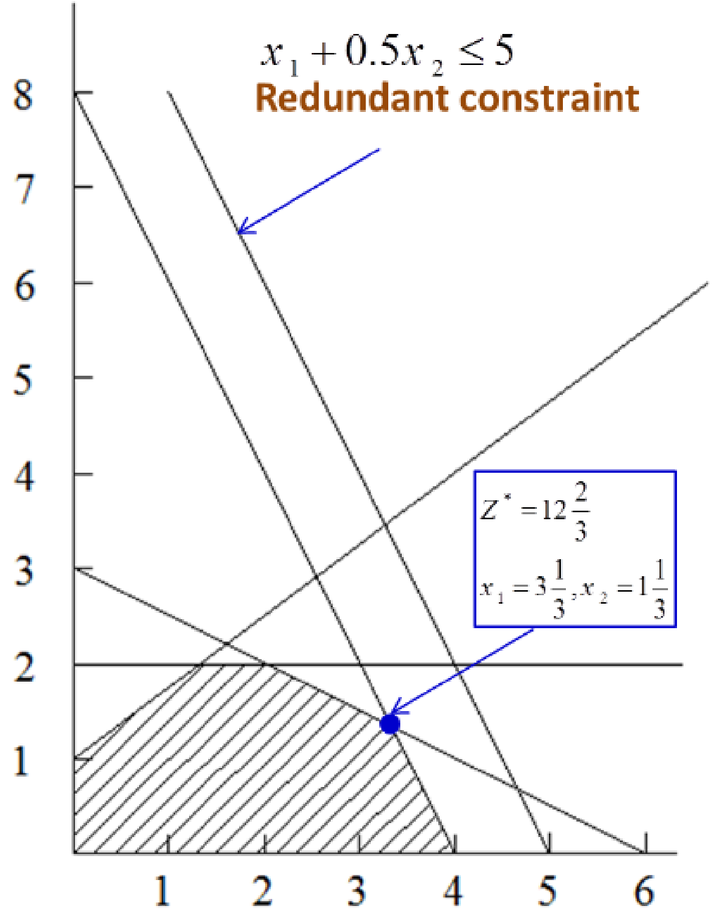
## Add a new limit

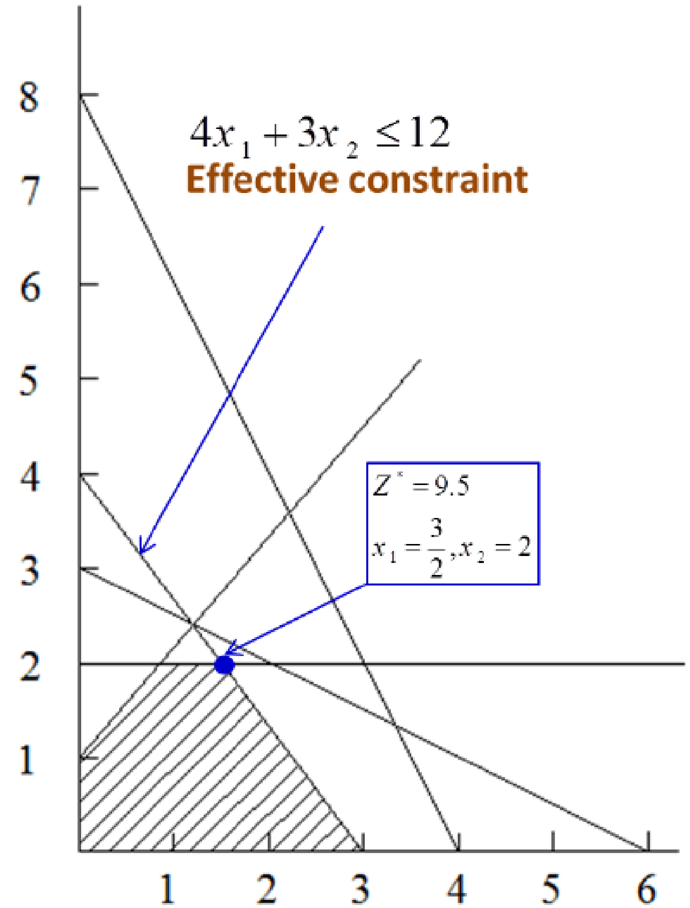
### 1) Adding a redundant constraint

As stated earlier, a redundant constraint is a constraint whose presence or absence has no effect on the feasible region and consequently has no optimal solution. According to the previous example, as shown in the figure below, if a restriction of $x_1+0.5x_2<=5$ is added to the previous four constraints, it will not affect the feasible region and the optimal solution remains optimal.

# Add a new limit

## 2) Addition of an effective constraint



# Parametric linear programming

# Systematic change of parameters cj

## Systematic change of parameters $c_j$

Consider the following objective function:

$$Z = \sum_{j=1}^{n} c_j x_j$$

In parametric programming, the above objective function is replaced by the following function.

$$Z(\theta) = \sum_{j=1}^{n} (c_j + \alpha_j \theta) x_j$$

# Systematic change of parameters cj

**Example:**

$$Max \quad Z = 3x_1 + 5x_2$$

$s.t.$

$$
\begin{array}{lllll}
(1) & x_1 & + x_3 & & = 4 \\
(2) & 2x_2 & & + x_4 & = 12 \\
(3) & 3x_1 + 2x_2 & & + x_5 & = 18
\end{array}
$$

$$x_i \geq 0 \quad i = 1, \dots, 5.$$

**Solution:**

Consider the value $\alpha_1 = 2$ and $\alpha_2 = -1$. Therefore, the objective function is as follows.

$$Z(\theta) = (3 + 2\theta)x_1 + (5 - \theta)x_2$$

# Systematic change of parameters cj

We start from the final simplex table with $\theta = 0$, the objective function becomes as follows.

$$Z + 1.5x_4 + x_5 = 36$$

We add the changes of the objective function to the left side of the objective function, which is as follows.

$$Z - 2\theta x_1 + \theta x_2 + 1.5x_4 + x_5 = 36$$

$$Z + (1.5 - \frac{7}{6}\theta)x_4 + (1 + \frac{2}{3}\theta)x_5 = 36 - 2\theta$$

$$1.5 - \frac{7}{6}\theta \geq 0 \rightarrow 0 \leq \theta \leq \frac{9}{7} \qquad 1 + \frac{2}{3}\theta \geq 0 \rightarrow 0 \leq \theta$$

# Systematic change of parameters cj

| Range of $\theta$ | Basic variable | Row | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{9}{7} \leq \theta \leq 5$ | Z | 0 | 1 | 0 | 0 | $\frac{-9+7\theta}{2}$ | 0 | $\frac{5-\theta}{2}$ | $27+5\theta$ |
| | $X_4$ | 1 | 0 | 0 | 0 | 3 | 1 | -1 | 6 |
| | $X_2$ | 2 | 0 | 0 | 1 | -1.5 | 0 | 0.5 | 3 |
| | $X_1$ | 3 | 0 | 1 | 0 | 1 | 0 | 0 | 4 |

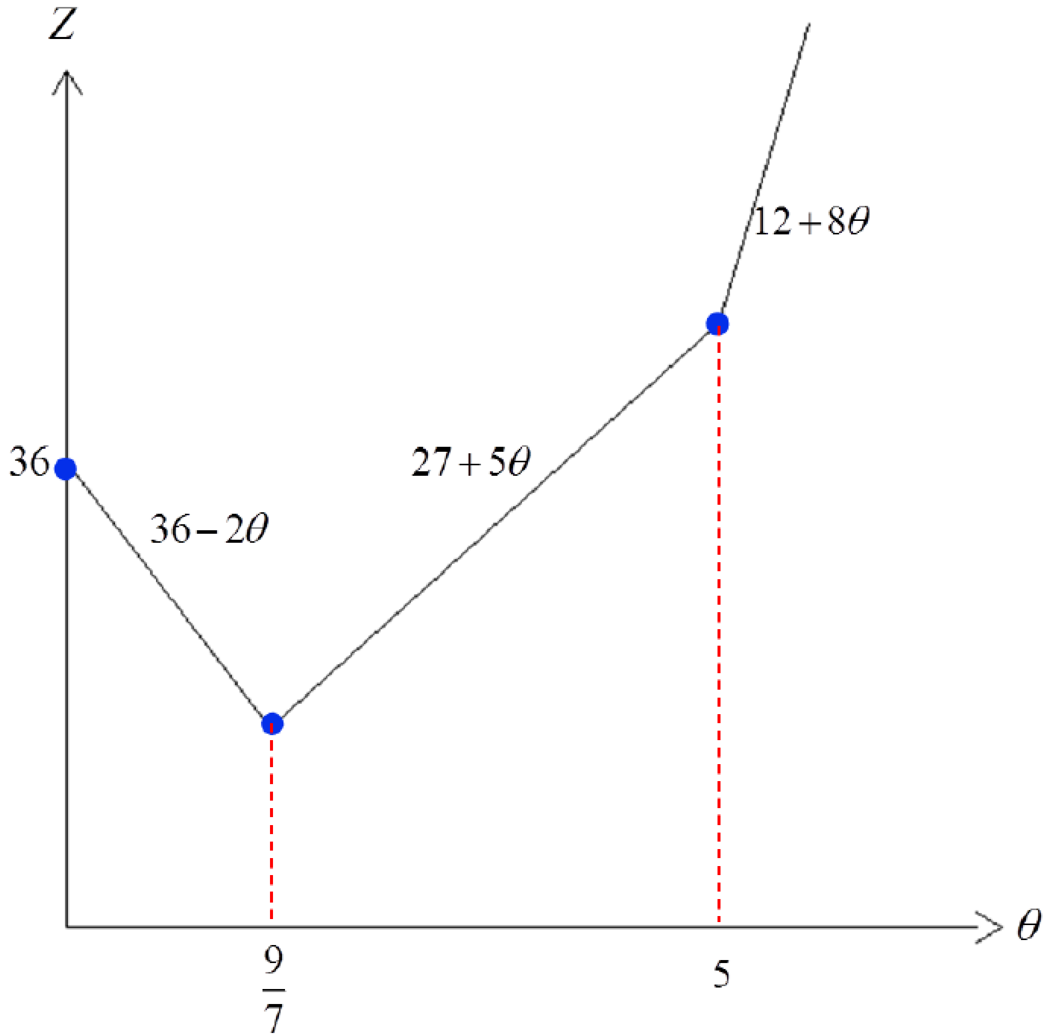
# Systematic change of parameters cj

| Range of $\theta$ | Basic variable | Row | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $\theta \geq 5$ | Z | 0 | 1 | 0 | $-5+\theta$ | $3+2\theta$ | 0 | 0 | $12+8\theta$ |
| | $X_4$ | 1 | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $X_5$ | 2 | 0 | 0 | 2 | -3 | 0 | 1 | 6 |
| | $X_1$ | 3 | 0 | 1 | 0 | 1 | 0 | 0 | 4 |

# Systematic change of parameters cj

| Range of $\theta$ | Basic variable | Row | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $0 \le \theta \le \frac{9}{7}$ | Z | 0 | 1 | 0 | 0 | 0 | $\frac{9-7\theta}{6}$ | $\frac{3+2\theta}{3}$ | $36-2\theta$ |
| | $X_3$ | 1 | 0 | 0 | 0 | 1 | 0.33 | -0.5 | 2 |
| | $X_2$ | 2 | 0 | 0 | 1 | 0 | 0.5 | 0 | 6 |
| | $X_1$ | 3 | 0 | 1 | 0 | 0 | -0.33 | 0.33 | 2 |
| $\frac{9}{7} \le \theta \le 5$ | Z | 0 | 1 | 0 | 0 | $\frac{-9+7\theta}{2}$ | 0 | $\frac{5-\theta}{2}$ | $27+5\theta$ |
| | $X_4$ | 1 | 0 | 0 | 0 | 3 | 1 | -1 | 6 |
| | $X_2$ | 2 | 0 | 0 | 1 | -1.5 | 0 | 0.5 | 3 |
| | $X_1$ | 3 | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $\theta \ge 5$ | Z | 0 | 1 | 0 | $-5+\theta$ | $3+2\theta$ | 0 | 0 | $12+8\theta$ |
| | $X_4$ | 1 | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $X_5$ | 2 | 0 | 0 | 2 | -3 | 0 | 1 | 6 |
| | $X_1$ | 3 | 0 | 1 | 0 | 1 | 0 | 0 | 4 |

# Systematic change of parameters cj



# Systematic change of parameters bi

## Systematic changes of the right parameters ($b_i$)

In this case, $b_i$ is replaced by $b_i + \alpha_i \theta$, where $\alpha_i$ are fixed data. Therefore, the problem is as follows.

$$Max\ \ Z(\theta) = \sum_{j=1}^{n} c_j x_j$$

$$s.t.$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i + \alpha_i \theta \quad i = 1, \ldots, m$$

$$x_j \ge 0 \quad j = 1, \ldots, n.$$

# Systematic change of parameters bi

**Example:**

Use the parametric linear programming procedure to perform systematic changes in $b_i$ and obtain the optimal solution of the following problem as a function of $\theta$ for $0 \le \theta \le 25$.

$$Max \quad Z(\theta) = 2x_1 + x_2$$

$$s.t.$$

$$(1) \qquad x_1 \qquad \le 10 + 2\theta$$

$$(2) \qquad x_1 + x_2 \le 25 - \theta$$

$$(3) \qquad x_2 \le 10 + 2\theta$$

$$x_i \ge 0 \quad i = 1, 2.$$

# Systematic change of parameters bi

| Basic variable | Row | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | -2 | -2 | 0 | 0 | 0 | 0 |
| $X_3$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_4$ | 2 | 0 | 1 | 1 | 0 | 1 | 0 | $25-\theta$ |
| $X_5$ | 3 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |
| Z | 0 | 1 | 0 | -1 | 2 | 0 | 0 | $20+4\theta$ |
| $X_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_4$ | 2 | 0 | 0 | 1 | -1 | 1 | 0 | $15-3\theta$ |
| $X_5$ | 3 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |
| Z | 0 | 1 | 0 | 0 | 2 | 0 | 1 | $30+6\theta$ |
| $X_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_4$ | 2 | 0 | 0 | 0 | -1 | 1 | -1 | $5-5\theta$ |
| $X_2$ | 3 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |

| Basic variable | Row | Z | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | RHS |
|---|---|---|---|---|---|---|---|---|
| Z | 0 | 1 | -2 | -2 | 0 | 0 | 0 | 0 |
| $X_3$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_4$ | 2 | 0 | 1 | 1 | 0 | 1 | 0 | $25-\theta$ |
| $X_5$ | 3 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |
| Z | 0 | 1 | 0 | -1 | 2 | 0 | 0 | $20+4\theta$ |
| $X_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_4$ | 2 | 0 | 0 | 1 | -1 | 1 | 0 | $15-3\theta$ |
| $X_5$ | 3 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |
| Z | 0 | 1 | 0 | 0 | 2 | 0 | 1 | $30+6\theta$ |
| $X_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_4$ | 2 | 0 | 0 | 0 | -1 | 1 | -1 | $5-5\theta$ |
| $X_2$ | 3 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |
| Z | 0 | 1 | 0 | 0 | 1 | 1 | 0 | $35+\theta$ |
| $X_1$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | $10+2\theta$ |
| $X_5$ | 2 | 0 | 0 | 0 | 1 | -1 | 1 | $5\theta-5$ |
| $X_2$ | 3 | 0 | 0 | 1 | -1 | 1 | 0 | $15-3\theta$ |
| Z | 0 | 1 | 0 | 1 | 0 | 2 | 0 | $50-2\theta$ |
| $X_1$ | 1 | 0 | 1 | 1 | 0 | 1 | 0 | $25-\theta$ |
| $X_5$ | 2 | 0 | 0 | 1 | 0 | 0 | 1 | $10+2\theta$ |
| $X_3$ | 3 | 0 | 0 | -1 | 1 | -1 | 0 | $3\theta-15$ |

# Systematic change of parameters bi

The results are summarized in the table below.

| $\theta$ | $(x_1^*, x_2^*)$ | $Z^*(\theta)$ |
|---|---|---|
| $0 \le \theta \le 1$ | $(10+2\theta, 10+2\theta)$ | $30+6\theta$ |
| $1 \le \theta \le 5$ | $(10+2\theta, 15-3\theta)$ | $35+\theta$ |
| $5 \le \theta \le 25$ | $(25-\theta, 0)$ | $50-2\theta$ |

# Exercises

# Thanks

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